BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

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VOLUME 64, NUMBER 4

JULY, 1958

(Whole No. 655)

CONTENTS

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND PROVIDENCE, R.I.

The annual dues of members of the Society (see this BULLETIN, November, 1956, pp. 641–642) are payable to the Treasurer of the Society, A. E. Meder, Jr., 190 Hope Street, Providence 6, R. I.

The BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY is published bimonthly, in January, March, May, July, September, and November. Price per annual volume is $7.00. Subscriptions, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450 Ahnaip Street, Menasha, Wis., or 190 Hope Street, Providence 6, R. I.

Entered as second class matter April 24, 1926, at the post office at Menasha, Wis. under the act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in the act of February 28, 1925, embodied in paragraph 4, section 538. P.L. and R., authorized May 9, 1935.

EVOLUTION BY MUTATION[1]

H. J. MULLER

It is not possible for me to represent the high tradition of Josiah Willard Gibbs by offering you a mathematical treatment. Nevertheless, the subject of biological evolution and its mechanism must be of great interest to yourselves, as the most exemplary products of its operation. Perhaps, then, our reconnaissance flight over these biological jungles, and our attempts to measure certain aspects of them, may serve to entice some of you or, through you, some of those with whom your influence counts, into bringing your higher powered mental tools to bear in the more effective and more elegant mapping and analysis of this territory. If so, my intention to inveigle you into it will have been successfully accomplished.

To those philosophers who declare "I think, therefore I am," their own existence seems the one complete certainty. To others, it does not seem so certain that they do think, nor even that they produce a significant imprint on reality in general. It is, however, evident that they, along with all things living, if they do exist, are utter improbabilities, far less plausible than any other phenomena that have been encountered.

Herein we shall attempt to assess how fantastically unlikely we and our fellow creatures are, and by what means such preposterous anomalies could have come about. The old-time philosopher still insists that such extravagances of organization could have arisen only by design, inasmuch as accident cannot be expected to convert itself into order. However, a dispassionate examination of the rules of this game of life should throw some light on the question of how such a massive compounding of improbabilities may have taken place.

1. **The genetic alphabet.** Studies in Mendelian heredity, supplemented by microscopic observations, gave evidence some half century ago that at the core of our being, and of that of every living thing, there is a remarkable material, that is particulate, exceedingly constant in its parts, subject to orderly mosaic rearrangements, and in a sense self-multiplying. All this was shown by the kaleidoscopic, yet statistically predictable effects it gave on being transmitted and

The thirty-first Josiah Willard Gibbs Lecture, delivered at Cincinnati, Ohio, on January 28, 1958, under the auspices of the American Mathematical Society; received by the editors March 14, 1958.

[1] This article is Contribution No. 649 of the Department of Zoology, Indiana University.

multiplied from generation to generation in the form of diverse combinations. The term "genes" was applied to the regularly recombining parts of this mosaic. Moreover, the fact was established on the basis of the quantitative relations which were observed in the recombinations of these genes, that they are strung together in a single-file arrangement, like the links in a chain, so as to form the microscopically visible filaments called chromosomes [1].

It further became clear that despite the constancy of the individual genes they are separately subject to rare, sudden changes, or "mutations," from one stable state to another. This is proved by the changed effects on the descendants that inherit them after such an occurrence. For these descendants then constitute exceptions to the original predictions. They are, potentially, the seed of new, although usually only slightly new, forms of life.

The most unique characteristic of these genes has long been realized to lie in the fact that, after a mutation has occurred, the gene in its changed form, on reproducing, gives rise to daughter genes that incorporate its new feature. That is, the mechanism of the gene's self-reproduction is such as to result in the perpetuation and, if circumstances permit, the multiplication of the deviant type. As has been pointed out elsewhere [2], it is the possession by the gene of this faculty of self-copying, of a kind that is capable of being retained despite changes in that gene's own composition, that causes the gene to serve as the basis of evolution. And the enormous complications to which evolution may go are made possible by the fact that these changes in genes and in groups of genes can become accumulated to a virtually unlimited extent, without entailing the loss of the genes' self-copying faculty. Moreover, it has become clear that, as had been surmised, the self-copying involves an attachment, next to each characteristic component of the gene, of a particle of corresponding[2] type that had been floating about in the medium surrounding the gene. In this way there becomes pieced together next to each gene a replica of itself, that is, a structure having the same internal pattern. A mutation consists of a permutation in this pattern.

Through the brilliant recent theory of Watson and Crick [3], backed by strong evidence from work of Benzer, Hershey, Stanley and many others, it has been virtually proved that the components in question are nucleotides, combinations of phosphoric acid, a simple

[2] As will be seen in what follows, however, the "corresponding" type here turns out to be a complementary one, rather than an identical one as had been first assumed. But since the gene contains pairs of complements to begin with this process works out to give a product that is identical with its producer after all.

sugar, and a nitrogenous base, the whole having a molecular weight of about 300. In the gene there are only 4 types of these nucleotides, that we may here call A, B, C and D. The gene consists basically of these nucleotides polymerized into the form (termed DNA) of a pair of relatively long parallel but coiled chains, of which the nucleotides form the links. In any such pair of chains A is always complementary to B and C to D, in such a way that A in one chain regularly has B lying opposite to it in the other chain, and C has D opposite to it. These opposite, or rather, complementary components form effective cross-unions with one another, and not with the other types of nucleotides, by means of hydrogen bonds. It is this fact that explains their selectivity in attaching to themselves only appropriate (complementary) particles derived from the medium, during the process of gene reproduction.

Now, although there are only two types of nucleotide-*pairs*, these amount to four types so far as the gene is concerned. For their arrangement within the gene is different according to which of the two members of a pair of nucleotides lies in a given member of the pair of chains. Hence, unless there are additional features that we do not yet know about, we could specify the entire composition of a gene through the use of four letters, A to D, setting them down in line, as in a word, in the order in which they occur in either one of the two members of the double chain. As yet, we are far from knowing this order in any case. But there is reason to infer that a gene-word is composed of thousands, even tens of thousands, of "letters."

A mutation, on this scheme of representation, consists in the substitution, loss, or insertion, of one or more of these same letters. Benzer's work [4] may be taken as indicating that only one letter, or nucleotide pair, is usually involved, but that at times a whole block of them may be inverted *in situ*, lost, or inserted. This same principle of what may be called point and line mutation has long been known to hold, on a far larger scale of magnitude, in the case of those greater chains, the chromosomes, the links of which are whole genes, some hundreds or thousands of them per chromosome.

2. **A measure of our own improbability.** We are now in a position to make some first estimates of the degree of improbability represented by our own genetic material. The total mass of nucleotide material, or DNA, contained in one set of human chromosomes, such as would be found in a human sperm or egg nucleus just before they united in fertilization, is approximately 4×10^{-12} of a gram. Since the mass of one pair of nucleotides is about 10^{-21} of a gram there

must therefore be about 4×10^9 nucleotide pairs in the chromosome set.

It is not certain that in higher animals the gene string, as we call it, contains only *one* double chain of nucleotides, but there is morphological as well as autoradiographic evidence that this is the case in bacterial viruses, and autoradiographic evidence in some higher plant material also. Moreover, the way that mutant genes have been observed to express themselves in some higher forms after a mutation has occurred, that is, the fact that in some cases all, in some cases about half of the cells descended from the cell in which the mutation has occurred may receive a replica of the mutant gene, indicates that this gene had not been in the form of more than two parallel threads. It therefore appears highly probable that even in man the genetic material of the sperm cell is in the form of unreplicated, merely double, chains of nucleotides.

This would lead us to conclude that all human gene strings of one chromosome set taken together contain some 4×10^9 nucleotide pairs arranged in one double line. It is possible some of the nucleotides are not in this line and are nongenetic, as Levinthal's preliminary results on bacterial viruses [5] had indicated to be the case in them. However, certain more recent findings have raised questions concerning this interpretation in the viruses, and the higher plant studies by Taylor et al. [6] have given grounds for considering virtually all the chromosomal DNA in them to be genetic.[3] This is a matter that the application of autoradiography to higher forms should soon give definite information about. Meanwhile, it will here be assumed that the number of genetic nucleotide pairs arranged linearly in a human chromosome set is the full number, 4×10^9, present in a human sperm cell.

Inasmuch as for each nucleotide pair there is a choice of 4 possible forms (representable as A, B, C or D) in a given member of the double chain, it is evident that the number of possible permutations of these four forms, in a line containing four billion of them, is four to the four billionth power or approximately $10^{2,400,000,000}$. It is true that this number should be reduced by dividing it by the number of permutations that would be possible among the 23 chromosomes and, more important, among all (some 10,000 to 40,000) entire genes, on the dubious supposition that most of these permutations would leave the genes' effects substantially unchanged. Nevertheless, on

[3] At the time of the lecture the reports giving the most recent evidence had not yet come to hand and the conservative assumption was therefore made that in man only 40% of the chromosomal DNA is genetic. Thus, in the text that follows, the figures are correspondingly higher than those that were presented orally.

making the maximum possible estimate for the magnitude of this divisor, a "mere" $10^{270,000}$ at most, we find the size of our exponent reduced by an amount that is entirely insignificant, in terms relative to its own size, and we may therefore feel justified in settling on the above approximation. Now, since any given individual chromosome set represents but one combination we may say that the "chance" of its occurrence is the reciprocal of this number, or $10^{-2,400,000,000}$.

It should be recognized that this figure may give an exaggerated impression of our uniqueness, since we do not know whether many nucleotide substitutions might be made that would have no effect, or virtually none, on the resulting organism. Moreover, many of them have such relatively slight effects as we see differentiating the persons about us. As against this consideration, however, there are grounds for inferring that losses of nucleotides, or of blocks of them, from the chain, occur as much more frequent accidents than do gains (that is, insertions) of them, so that there is a tendency for unnecessary elements to be eliminated eventually. Let us then take our approximation at its face value and try to arrive at a working idea of its magnitude by comparison with something familiar to us in everyday life. It can be estimated that a large, finely printed edition of Webster's Unabridged Dictionary contains about thirty million letters. If, then, we used only the letters *A*, *B*, *C* and *D*, to represent the four nucleotide pairs, we could represent the entire arrangement of them in a single human sperm or (pre-fertilization) egg nucleus by the use of about 133 volumes, each of the size and fineness of print of this dictionary.

Here, presumably, we should have the entire genetic specification for a man, at least so far as his inheritance from one of his parents was concerned, and another 133 volumes would give that from his other parent. With the know-how (as yet not in sight) of how to string nucleotides together indefinitely as desired, and to give them the right wrappings, we should then be able to insert them into an egg from which its own nucleus had previously been removed and thus, after enormous labor, helped perhaps by automation, to produce a man as much like the one who had furnished the specifications as if he were an identical twin. Or we might incorporate alterations in him to order.

If instead of representing each nucleotide-pair separately by *A*, *B*, *C* or *D*, we utilized the entire English alphabet of 26 letters, plus half a dozen distinctive Cyrillic letters to make 32, and if we then allowed the letters to be either in lower case or capitalized, either in the slanting italic or vertical roman style, and either heavy faced or fine, thus gaining 256 distinguishable characters, we could allow each character

to represent a group of four nucleotide pairs instead of just one pair (since for each group of 4 there would be 256 possibilities). We could thereby reduce our 133 volumes to 33. We could also greatly condense the representation of the inheritance from the second parent by designating only those items of it that differed from the corresponding items derived from the first parent, and by inserting these modifications, with appropriate punctuation, at the points in question, in line with the specifications of the first parent's contributions. By then transferring our perhaps 34 great volumes to especially thin microfilm we should be able to get our coded *homo* into the space of one volume having the outer dimensions of a scientific handbook. However, we may recall that, by contrast, the actual nucleotide material of a human or other mammalian germ cell, when mature, would occupy only about four cubic microns, of weight 4×10^{-12} grams.

3. **An alternative measure.** There is an older method of estimating our improbability [7] that can now be brought more nearly up to date. Both observations and general considerations make it likely that, very conservatively, not one among 100 mutational changes with a presently detectible effect on the organism is conducive to its survival or fertility and thereby favors the multiplication and establishment in the species of the given mutant type. Moreover, any accidental accumulation of smaller changes that together resulted in as much deviation from the original type as those here in question would have a similarly small chance of being advantageous. The reason for this prevailingly detrimental character of mutations is of course the fact that there are far more ways of damaging the workings of an already elaborate and well constructed organization than there are of improving it even further. This situation is analogous to that of the second law of thermodynamics. In the latter case the energy of particles subject to random motion tends to become dissipated because of the fact that there are more directions and amplitudes of movement by which the energy can be scattered than those by which it can be concentrated. So, in general, there are more types and degrees of change that are disorganizing in relation to the production of a specific result (in the case of living things, their multiplication) than those that are further organizing.

Simplifying the situation by first considering only nonsexually reproducing organisms, and taking the conservative estimate of 99 detrimental to 1 advantageous change of perceptible degree, it follows that, on the average, the mutant type must multiply at least a hundred fold after each advantageous mutation if evolution is to

continue. This multiplication is necessary to make the individuals, or rather, the lines of descent, of the advantageous type numerous enough to allow just one of these hundred lines to give rise to a second advantageous mutation, added to the first one. And so on after that, for each successive advantageous mutation that is accumulated in the same line of descent, there must on the average be a further multiplication of at least a hundred times.

In the meantime, the individuals with the disadvantageous mutations, and in some cases those of the original type also, will have tended to die out, thus making room for the line having the concentration of favorable changes. For the latter, however, the rate of multiplication will have averaged so high that, had this same rate characterized the entire population, their final number would have been at least 100 to a power equal to the number of successive advantageous mutations that accumulated in the favored line. Thus, for 3 beneficial mutations this number would have been 100^3 (or 10^6) and for 100 mutations, 100^{100} (or 10^{200}).

These figures are relevant to our inquiry into the degree of our own improbability. For suppose that, instead of having started with just one individual which, in its more favored lines of descent at least, was able to multiply at a rate that would have given the number calculated, we had, instead, *started* with as many individuals, and therefore with as many lines of descent, as that imaginary final number that would have resulted from the equal multiplication of all lines at a rate as high as that of the favored lines. There need in that case have been no multiplication at all, or any ability to multiply, but only a persistence of the individuals, or of their single-file "lines of descent." In fact, even this persistence need have occurred only in the lines in which successive "favorable" changes happened to occur. Yet, given the same rate of "mutation" as before, we should on this system have ended up with just as many individuals having the maximum number of "favorable" changes as on the other system. For in both cases an equal number of individuals would have been provided, in each generation, in which disadvantageous (i.e. self-eliminating) mutations had not yet occurred, and in which "favorable" ones (i.e. those of types analogous to the mutations which would have favored multiplication had it been possible) could therefore have accumulated instead.

4. Differential multiplication as the extractor of the improbable. This "thought experiment" (to use the physicists' term) is, like most such experiments, fantastic, but illustrative of a principle. In this

case it shows not only the degree of improbability achieved by the succession of mutations in the favored lines but also the role played by the process of biological multiplication in allowing this degree of improbability to be achieved. Thus the scheme on which there was no multiplication shows that the individual that had accumulated 100 favorable mutations represented a combination of chances that could happen only once in 10^{200} trials. There is no possibility that this number of trials could ever, on our earth, have been achieved. Yet the process of multiplication, by being differential, that is, largely confined to the lines that continued, accidentally, to have the favorable mutations in them, succeeded in providing the opportunity for the realization of this degree of improbability.

Within the narrow confines of our world, this multiplication of the favored lines was able to occur only because space was left by the dying out of the other lines. That, then, was the role of selective elimination: to make room. But advantage could be taken of that room only by reason of the gene's faculty of reproducing itself, and thereby multiplying. And even this could not have resulted in evolution if the gene had not been so constituted as to reproduce its mutational changes also, in the process of reproducing itself. Evolutionary adaptation is thus the automatic result of the differential multiplication of mutations. And living things are so much more elaborately organized than nonliving ones because the gene's unique property of self-copying constitutes the basis for this differential multiplication of its changes.

Thus, on the primitive earth, after the myriad interactions of diverse substances, occurring in a medium of water and powered by high-potential discharges of photons and electrons, had resulted in the production of nucleotide molecules, and then attachments between some of them to form naked genes of the most rudimentary type, that fed on those that were free, their further evolution to produce their protoplasmic wrappings and finally all the complications of the intricately adapted organisms of today, followed from the pressure of their differentially multiplying mutations.

But we have not yet followed far enough in applying this method of estimating the degree of our improbability. This method, it may be recalled, proceeds by first estimating, conservatively, the probability that a given mutational step will be successful, and it then raises this figure to a power equal to the estimated number of such steps.

On reconsideration of the probability of success, which is the reciprocal of the number of mutations necessary, on the average, to

include one that is successful, it might at first sight seem that, for a given nucleotide-pair, the number of possible substitutional changes that would include one successful one should be no more than 3, since there are only 4 types among which to choose and one of these four types is already present. This inference, however, besides overlooking possible losses, insertions, and inversions, neglects the much more important fact that on the great majority of occasions any change at the given point would be disadvantageous. Usually there would be no possibility of any change at some *given* point in the nucleotide chain being advantageous until some change or combination of changes, of given kinds, had occurred elsewhere, that somehow upset previously attained adaptations. In other words, a group of successful steps accumulated over a period that, in terms of evolutionary time, is very long, would not have been successful if they had arisen in a radically different sequence from the actual one. This restriction explains why the chance of success for a change occurring at any given point, at any given time, can be less than 1 in 100 or even less than 1 in 10,000 despite the fact that the number of possible changes at that point is (if we exclude the comparatively rare cases of insertions and inversions) very limited.

Accepting, then, the very conservative figure of 1 in 100 for the chance of a given mutation being successful, what shall we assume for the exponent of this figure, that is, for the total number of successful mutations in the ancestry of a given higher organism? This total number may obviously be regarded as the product obtained by multiplying the number of successful mutations that have occurred per gene by the total number of genes. As for the number of past mutations per gene it is to be observed that, as was realized long ago, e.g. [7], each individual gene must be highly adapted and complicated, and have arrived at its present form through numerous steps. Knowing, today, that it contains thousands or tens of thousands of nucleotide pairs, we might estimate the number of steps per gene to have been as great as this or even much greater; that is, we might assume a past history of several or many substitutions of each pair. Nevertheless, we are, to remain on the conservative side, contenting ourselves at this point with the undoubtedly far too low figure of only 100 successful steps per gene.

In taking this figure we are bearing in mind the fact that in the distant past the genes were derived from one another, through rare accidents such as occasionally happen even today, whereby a block of them derived from one chain becomes inserted at some point into another chain. In consequence, many of the earlier mutational steps

occurred in genes that were common ancestors of several or many present-day genes, and our assumed number of 100 steps per gene refers to the total number of independently arisen mutations, averaged out per gene. Yet even considering this, the actual number of steps is more likely to have been many thousands than only 100 per gene, because there are grounds for inferring that gene numbers of the order of those at present existing were already attained more than half a billion years ago.

Taking now the number 10,000, derived from flies, as a minimum estimate for the number of different genes in a higher organism (despite the fact that the higher organism contains a far larger total number of nucleotides), we see that there must have been at least $100 \times 10{,}000$, that is, a million separate successful mutations in the ancestry. Applying this million as an exponent to 100 (our conservative figure for the reciprocal of the probability that a mutation will be advantageous) we then get $100^{1{,}000{,}000}$ (or $10^{2{,}000{,}000}$) as the total number of trials that would have been necessary, in the absence of multiplication and selection, to obtain one combination as well organized as our own or as that of some other advanced organism.

Although so much smaller than our other estimate of about $10^{2{,}400{,}000{,}000}$, based on the number of nucleotide pairs, the present more conservative number deserves some scrutiny, some comparison with more familiar things. In this connection we may ask, how much room would it have taken to contain this many combinations of genes at one time, in order that amongst them our own constitution might find a place as one of these random occurrences? A sphere having a diameter of six billion light years goes far beyond the most distant galaxies now detectable. For our present purposes, however, we shall call it, by a stretch of terminology, "the known universe." A little arithmetic will show that in this vast expanse there would be room, if they were all packed closely together, for about 6.25×10^{100} packets or skeins of nucleotide chains, such that each skein contained as many nucleotides as we have taken to exist in a mammalian sperm nucleus, namely, 4,000,000,000, the number that we previously found it necessary to employ 133 Webster's volumes to represent. Yet we see that this enormous number of packets, 6.25×10^{100}, is inordinately smaller than the number $10^{2{,}000{,}000}$, that on our more conservative estimate could be expected, as a random event, to include a packet with a composition as select as our own. And even if we had some science-fictionist's method of reducing the size of a genetic packet to that of a proton, we could still get only about 10^{128} of them into the known universe.

Suppose, now, that in order to attain our desired number we allowed each of these packets or genetic combinations to exist for only a millimicrosecond, that is, a billionth of a second, and then caused it to be replaced by a different combination, and so on every millimicrosecond in succession for six billion years, which is probably longer than the earth has existed. This would have allowed some 2×10^{26} changes and we should thereby be able to accommodate in the "known universe" during this period about 2×10^{154} genetic combinations. Let us institute next the radical procedure of allowing each of the evanescent proton-sized spaces thus obtained to be itself expanded to the size of our known universe, and to be granted a time-span of six billion years, within which it in turn became subdivided in both space and time just as we had previously subdivided our own known universe. The total number of genetic combinations that we could get in this way would now be the square of the previous number, and thus come to 4×10^{308}. But we should have to go on in this way, expanding protons into worlds and millimicroseconds into eons and then subdividing them as before, through about 14 cycles, before we attained the more conservative number, $10^{2,000,000}$, that we are seeking.

This result, then, may give us some glimmer of an image of how improbable we are. How right, then, in a short-sighted way, were those ostensible "savants" who, so they declared, found it "philosophically unsatisfying" to believe that they, or any other living things, had come about by accident. For what an unthinkable multitude of universes would have had to be searched through, before so improbable a combination of accidents as themselves could have been found. And yet, the near-magic faculty of multiplication by self-copying, possessed by the nucleotide chains, does give the opportunity for these most select combinations of accidents to arise. For the multiplication rate in *their* lines of descent *was* enough, had it been extended to all lines, to have produced that superlative number, of which our own combination formed just one unit. And after all, the persistence of the defective lines was not necessary for the outcome. In practical fact, on the contrary, their elimination was necessary.

5. **The role of sexual reproduction.** We may next inquire whether the period during which life has existed on the earth has been long enough to allow such a succession of multiplications as here required, that is, a one-hundred-fold multiplication occurring one million times in succession. Dividing these million steps among the three billion years or so during which fossil evidence indicates life to have existed

on the earth, we find 3000 years allowed, on the average, for each of the 100-fold multiplications. Now the number of generations occurring in every period of 3000 years has diminished from, potentially, millions, in stages corresponding to bacteria, to about 3000 (or 1 per year) for many of the lowlier many-celled forms, and then down to some 100 (or 1 in 30 years) in the case of modern man. At the same time, among many-celled forms, the potential amount of increase per generation has also diminished greatly. However, even modern man in America is now doubling his numbers every forty years, a rate which if continued would give a 100-fold increase in a mere 266 years. Of course, an advantageous mutant could seldom be expected to multiply so rapidly as this, relatively to the rest of the population. If, as seems reasonable, it had only a 1% reproductive advantage over the other individuals, it would require some 70 generations to achieve a doubling, and 465 generations for a 100-fold increase. This in man would occupy some 14,000 years. But since the human generation is so much longer than that which obtained in our ancestry until relatively recently, there was undoubtedly plenty of time for a million steps altogether.

We do become pinched for time, however, if we attempt, by this method, to squeeze in as many or more successful steps as our number of genetic nucleotide pairs, that is, some 4,000,000,000. For this would give only about a year, on the average, for each hundred-fold multiplication. If, as seems likely, each nucleotide pair has a history of several independent substitutions, and if by reason of the rarity of advantageous steps each period of multiplication requires an increase of 1000- to 100,000-fold rather than one of only 100-fold, then, as can readily be reckoned, each successful mutation would have had to double its numbers, on the average, every few days! This means that it would have undergone something like a 1% relative increase every hour.

Fortunately, the genes have found a way of meeting this evolutionary difficulty. Their answer is sex! Or rather, more precisely stated, it is sexual reproduction. The function of this arrangement is to expedite evolution by making it possible to obtain an accumulation of advantageous mutational steps without having the respective multiplications of these steps occur in series. They are allowed, instead, to occur in parallel, with concomitant interpenetration and combination of the respective lines of descent [8].

Let us first be clear concerning the basic genetic process involved in sexual reproduction. The act of fertilization that produces the child brings together two groups of chromosomes, or chains or genes,

of somewhat different ancestry. Although each of the two groups by itself comprises one virtually complete set of genes, there are some mutant genes present in each set. Now the mutant genes of one set are represented in the other set by a gene of the original type, or, more rarely, by a different mutant gene, lying at a corresponding position in a chain of that other set. At some time before the act of fertilization that results in the next generation—we shall call these the grandchildren,—the two sets of gene-chains line up parallel with one another, with their corresponding genes in apposition, a process called synapsis. In some viruses, at any rate, even the corresponding nucleotide-pairs lie in apposition at this stage. Following this synapsis, the apposed chains again separate, and they become distributed to different germ cells, each cell now receiving just one complete set instead of both of the sets that had come together.

The function of the coming together or synapsis is, for one thing, to accomplish an orderly separation, that insures each germ cell's receiving one complete set. But there is an even more important function in the synapsis. For, during their apposition, a considerable interchange of corresponding parts takes place among and between the chains of the two sets. Not only may a given germ cell thereby receive a whole chromosome (call it number 1), that had belonged to one set, and simultaneously another whole chromosome (2) of the other set, but many of the individual chromosomes that it receives are themselves mosaic. They are mosaic in the sense that their gene chain up to a certain gene, or nucleotide pair, has been derived from a given chromosome (call it number 3) of one of the sets, and from that point on has been derived from the corresponding chromosome (3) of the other set (see ref. [1]).

In higher organisms this interchange of chromosome parts, called crossing over, occurs by means of an actual breakage of the two corresponding gene-chains or chromosomes, one from each set, at the same point along their length, followed by the attachment of the left-hand part of one chain to the right-hand part of the other, and conversely between the other two parts. However, in the virus studied by Levinthal [5], the interchange is accomplished by the reproduction of the gene chains in such a way that one daughter chain is the daughter of one original chain up to a given point and of the other original chain from that point on. But in either case the outcome is the same. That is, each germ cell comes to contain, and bequeaths to the grandchild, just one complete set of genes, of which however certain ones trace back to the grandfather and the others to the grandmother on that germ cell's side. A grandchild, then, may receive

mutant genes from both sources at once, and it is able to transmit them together to its descendants. Remote descendants may thereby come to inherit this combination from both their parents, and to "breed true" for it, as we say.

Let us try to visualize mentally how this process expedites the accumulation of successful mutations. Suppose a horizontal row of dots at the top of a diagram represents a population at a given time, comprising n individuals, say 100,000. On the next line down are their descendants, averaging 1 apiece and therefore also n. If one *favorable* (i.e. advantageous) mutation occurs among f individuals, say 10,000, the number of favorable mutations in the first generation is n/f or 10. Suppose the reproductive advantage, r, averages 1%, in that 100 favorable mutants of this kind would in this setting tend to produce 101 offspring, as compared with 100 offspring from 100 non-mutants that did not have this competition. With this linear logarithmic increase it will take about 70 generations, on the average, before the number of these favorable mutants that arose in the first generation, 10, had been doubled to make 20. It is true, however, that the number of generations actually taken by the doubling would have a relatively high error. Moreover, many of the mutants would die out accidentally along the way while, as if to make up for these, there would be a much higher than average multiplication of some of the others. Here, however, we need consider only the averages. We may then ask the question: how long would it be before two favorable mutations had been accumulated in the same individual?

We shall take first the simpler case, that of organisms that do not reproduce sexually. In a case of this kind a second favorable mutation may be expected to arise in the same line of descent as that already containing one favorable mutation at such a time, on the average, as f individuals had been produced altogether, in that "line." That is, we do not have to wait until there are f (or 10,000) individuals of that line in one given generation but only until their sum in all generations has become f. The number of generations, g, required to attain this sum, f, is readily obtained, since g in this case represents the number of terms in a factorial series beginning with 1, in which each term is $(1+r)$ times the preceding term, and in which the sum of the terms is f. (Here $g=[\log(1+rf)/\log(1+r)]-1$.) Where, as in our numerical example, $f=10{,}000$ and $r=.01$, g, the number of generations required to accumulate 1 favorable mutation in addition to the first one, turns out to be approximately 464. Moreover, the number of generations, g_m, required for the accumulation of any given number, m, of such additional mutations, is simply mg (e.g. in our

example 9280 generations would be required for 20 of them). Or, conversely, $m = g_m/g$.

We may now compare this result with that in a sexually intermixing population having otherwise the same characteristics. In this case, by the time the generation g (or 464) is reached that in the asexual population would on the average have been necessary before a second favorable mutation was superimposed on the first one, there would have been a total population of gn individuals produced, and among all these there would have been gn/f favorable mutations. Now if we are dealing with a long period, of the order of tens of thousands of generations, such as those usually involved in considerations of "macro-evolution," we can ignore the length of time needed for any two favorable mutations of independent origin to become recombined so as to be present together in the same individual, or germ cell. For, in a relatively small fraction of such a period, the great majority even of mutations with as low an advantage as $r = .001$ would have had time to spread over practically the whole population. In so doing, these different mutant genes would have undergone the recombinations necessary to bind them together, that is, to incorporate them into the same chromosome sets.

Accordingly, in the sexual populations, virtually all of the gn/f favorable mutations arising during each period g (that in the asexual population allows just one more favorable mutation to accumulate) will have the opportunity of being eventually accumulated within the same descendants. Thus in any extended period represented as a large multiple of g, such as mg, the individuals of the sexual line can accumulate some mgn/f favorable mutations while those of the asexual line accumulate only m of them. That is, over a long period the speed of evolution in the sexual lines will be gn/f times that in the asexual lines. Even when a more unfavorable combination of numerical values is assigned to these terms than would often occur in practice (as when g is taken as only 10^2, n as only 10^6 and f as 10^6) this ratio is considerable (in this case 100). It may be inferred then that, ordinarily, sexuality increases the speed of evolution by a factor of many thousands and in some cases even millions. This enormous acceleration explains how it has been possible for several or many billions of mutations to have been accumulated by natural selection in the course of 3 billion years.

There is one factor that tends to make the situation even worse than this for the asexual as compared with the sexual population. This lies in the fact that in the former the favorable lines usually enter into an increasingly restrictive competition with one another,

thus reducing each other's selective advantage, whereas in the latter the formation of combinations of them tends increasingly to substitute cooperation for competition.

When a considerable period is under consideration, comprising tens of thousands of generations, the effective population number, n, to be used in the above formula, is that of practically the entire area between the parts of which any intermixing occurs, rather than the average number present within the partly isolated local groups usually dealt with in population-genetic studies. For in the course of the long period in question sufficient migration usually takes place between these groups to allow locally multiplied genes that would have a favorable influence in the group as a whole to become spread throughout the area. Because of the prodigious size attained, for many species, by the population of the all-inclusive area, the speed of their evolution becomes, over a long period, enormously enhanced by sexual reproduction. It should therefore be no matter for surprise that, having once arisen in primitive organisms, this procedure should have been retained by the great majority of species.

The above outlined mode of action whereby sexual reproduction allows evolution to proceed more rapidly has sometimes been misunderstood. According to this misconception, one of the ways in which sexual reproduction aids evolution is by allowing combinations to become formed and tried out, the individual genes of which would not have been advantageous in the general population but which, taken together, constitute a favorable complex. Undoubtedly there are many cases of such genes and they do play a significant role in evolution. It is to be noted, however, that in these cases there would be no greater opportunity for the lucky combination to arise in a sexual than in an asexual population. Only mutant genes that are advantageous in at least a local population, and thereby spread within it, have a greater chance of forming combinations through crossing than through successive mutations that occur in series as in asexual organisms. The fact that sexual reproduction is so widespread therefore attests to the great evolutionary importance of genes whose favorable effect does *not* depend on their presence in combination with other special mutant genes of independent origin. In other words, it attests to the prevalence of so-called additive effects of genes as opposed to complementary ones.

Sexuality got a far earlier start in evolution than was realized until, some thirteen years ago, the recombination process was discovered in bacterial viruses by Delbrück and soon afterwards in bacteria by Lederberg. Moreover, at about the same time, the idea arose [2c] and was later shown to be correct, that the so-called transforma-

tion of one line of bacteria by application of nucleotide chains from another is really a modified instance of the sexual recombination process.

It is true that some groups of organisms, including even higher organisms, in every period of the earth's history, have dispensed with sexual reproduction in fact or in effect, and that this has given them the considerable temporary advantage of being able to multiply without having to wait for the nuisance of finding and pairing with one another first. But these can have only a transitory splurge and are doomed to fall behind in the long evolutionary race and to disappear. They furnish an illustration of the shortsightedness, the opportunism, of natural selection. The stem forms of evolution, from which the organisms of later periods will be derived, are those that pay their tax to sexuality and are repaid in novel developments.

That even forms which have not undergone outwardly appreciable evolution for scores or hundreds of millions of years, such as some molluscs, have for the most part retained sexual reproduction, testifies to the continuing value for them of evolutionary adjustment of less tangible kinds. Among such adjustments are to be classed relatively temporary ecological adaptations, often largely invisible, that bring them into line with shifting conditions of their physical, chemical and biological environments. Sexual reproduction allows much prompter genetic accommodation of this kind. Another important group of changes in seemingly unchanging species are those, further discussed in §7, that result in improved regulatory responses, including both more accurate, wider range, and more versatile stabilizing mechanisms, and in the improvement of means of exploiting the environment. Such evolution is to a considerable extent cryptic, i.e. beneath the surface open to our present means of observation, for there are undoubtedly far more reactions of this kind in any organism than those of which we are aware. Progress in such directions must often involve selectional steps that individually confer only a minute advantage. Thus the selective pressure, being of only third or fourth order magnitude, requires, even with the aid of sexual reproduction, a very prolonged period for the achievement of important results. Nevertheless, taken together, these results, which the asexual species would be far slower still in attaining, may eventually be of decisive significance in the competition for survival.

6. The importance of localized evolutionary experiments. A factor materially affecting the establishment of advantageous mutations is the degree of subdivision of the species into semi-isolated groups. This factor, which has been treated mathematically by Sewall Wright

in numerous publications, is of especial importance in the case of genes of the type referred to in the 4th paragraph preceding, i.e. those that have a net favorable effect only as special combinations and not on the average when acting in connection with the genetic constitution of the population in general. We have seen that in such cases sexual recombination, operating widely throughout a large population, does not facilitate the formation and spread of these combinations. Yet in the long run such combinations are often of great importance, and if established may act as turning points that allow evolution to proceed in a new direction. As Wright has clearly shown, populational subdivision can greatly facilitate the establishment of these combinations.

There are two ways in which such subdivision can have this effect. For one thing, the number, n_l, of individuals in the local group is often so small as to allow some mutant genes that by themselves, even in connection with the genetic constitution of that local group, have no favorable effect, to become relatively numerous, merely as a result of the large random fluctuations to which small numbers are subject, a process termed "drift" by Wright. In some of these cases two or more such mutant genes which would be favorable only, or mainly, when in combination with one another, will thereby accidentally get the opportunity of being present together. This could of course happen just as well in asexual reproduction also. Having now become, as it were, superposed, their favorable joint action will come into play, so as to promote their spread. Under the circumstances of sexual reproduction, they can then spread much more rapidly and surely in the small group than if they were subject to the greater dissipation from one another that a larger group would entail. (A large asexual population, however, is not subject to this limitation.) Finally, by gradually diffusing out from the small group into its neighbors, and sometimes by the gradual advance of the local group as a whole into ever larger territories by competition, groupwise, with its less well equipped neighbors, these combinations can then proceed to "take over" in the general population, n.

The other and probably more important process depends on the many different selectional conditions to which the different local populations are subject. These tend to make some genes favorable from the start in a given local population that would not be favorable by themselves in the population as a whole. Both the peculiarities of the local environment (including the biological environment consisting of other species) and also the peculiarities of the genetic content of the local population itself (that arise as a result both of drift and of this very process of local selection) constitute important factors

in the causation of these selectional differences. In consequence of them, some combinations of these locally advantageous genes can become "established" in the sub-population (this time with the aid of sexual reproduction), which *as combinations* though not as separate genes would have a selective advantage even in the larger population or the species as a whole. And again, just as when such combinations had arisen through drift, these locally numerous combinations can then proceed gradually to spread throughout the species.

In both these ways, then, the subdivision of the large population into groups that are locally or temporarily more or less isolated from one another in reproduction allows the carrying out of numerous small-scale evolutionary experiments that would not have been permitted in the freely interbreeding or so-called "panmictic" large population. On the whole not as many evolutionary possibilities, nor as radical ones, are available for the reproductively undivided group. It is this latter type of population that mankind is rapidly approaching today.

The method of local experiments is not, however, the only way by which evolutionary corners can be turned and new directions embarked upon. For even to the large group new pathways may be afforded by alterations in conditions of living. These will oftener result from changes in the biological environment (that provided by other species) than in the inanimate environment, because the biological environment is so much more complex, diversified, and itself subject to change, than is the inanimate environment. The new pathways can also be presented, even to large relatively undivided species, when through the acquisition of given favorable mutations, or combinations of mutations that had been favorable even individually, the species acquires one or more faculties, or passes some threshold in the development of one or more faculties, that allows it to exploit a new mode of living.

Following any such turning of a corner there is likely to be a period of much faster evolution than before. For the longer a species has been selected for its old ways of life the harder it is to find new mutations that adapt it to these ways still better. On the other hand, for life carried out in a new way or (what amounts to much the same thing) under new conditions, many mutations that would previously have afforded little or no advantage will now be found helpful.

In such cases it is also much more likely to happen than before that the population, in its different parts, will find different methods of adapting to these new ways, and will find the new ways themselves to open up in diverse directions. Thereupon there will be a tendency not merely to faster evolution but also to a splitting of the

species into different lines. At first these lines will be isolated from one another in their reproduction mainly by geographic boundaries, but later genetic barriers (including genetically based physiological barriers) will arise between them as well.

The problem of how species split, in genetic terms, is one that is too ramified to permit of treatment here. It should be obvious, however, that the more numerous and the better isolated the local sub-populations are, the more such splitting is facilitated. It should also be observed that, the more any two sub-populations diverge genetically from one another, even in cases in which they retain great resemblances in their form and manner of functioning, the more likely they are to accumulate sets of genes that can no longer function effectively after having become mixed or recombined with one another [9]. We must recognize them as separate species after such mixing has become, in a state of nature, virtually impossible in consequence of these genetic incompatibilities.

7. **The development of stability and lability.** But the long course of evolution is by no means concerned only with the dramatic turning of corners and the multiple branching of pathways. Through the prolonged periods of seeming stasis there is, as noted in §5, a gradual genetic whittling away at structures and functions, an increasing refinement of them in adjustment to outer circumstances and to each other. Most of this is beneath the surface that is open to our present relatively crude means of inspection. Even at the turning of corners most of the individual mutations that succeed entail relatively small changes, since larger ones, despite being in some cases favorable in themselves, usually involve maladjustments of the already achieved delicate balances in the complicated interworkings of parts. Thus the other parts must gradually be changed correlatively, before a further change in the primary direction becomes profitable.

In consequence of these relationships the progression in a relatively new evolutionary direction is gradually enabled to go further and further. The appearance is thereby presented of an inner tendency to keep on varying genetically in the given direction rather than in other directions, a fiction denoted "orthogenesis," that has no sound basis in genetic reality.

It does remain true, however, that the organism because of peculiarities of its constitution is able to undergo genetic change much more rapidly in some directions than in others, and in some not at all. Moreover, this pattern of genetic inclinations is itself subject to change through mutation [10]. But these limitations are due largely to "canalizations" of its developmental and physiological processes,

if I may revert to my original meaning [7] of a term that has since then been used in diverse senses. Such inclinations, be it noted, can never force evolution to proceed in a given direction if the interests of the species are occasioning a selection that works in the opposite direction.

When evolution has not recently taken a radically new direction the visible alterations are, of course, still slower. Yet much is often going on beneath the surface, that may profoundly affect the embryological, the physiological, and the ecological reactions of the organism. One of the most important classes of changes in this category concerns itself with the achievement of ever greater stability of development and of operation for structures and functions that are regularly needed. As pointed out long ago [11] and further emphasized since that time [12], mutant genes must gradually have been selected that gave greater stability and dependability both to the organism itself, in its characteristics, a property that might be denoted as "phenotypic stability," and also those that gave greater stability to the genes themselves, protecting them from the action of agents that might otherwise produce mutations in them.

Too much of a digression would be required for an adequate treatment of this matter in the present article. Suffice it to say that, so far as both kinds of stability are concerned, there is ground for inferring that many different mechanisms have been adopted: in fact, whatever came to hand. Some of these have had the effect of making the reactions in question especially resistant to disturbing influences. Others have worked by counteracting those influences themselves, somewhere along the line. Still others have involved self-regulatory or what are now termed "cybernetic" processes. Moreover, there has been, in phenotypic regulation, the attainment of "factors of safety," or means of doing the same thing through reactions that are somewhat different, involving more or less alternative pathways that are opened up where needed. The term "homeostasis" has often been applied to all this stabilization. It should be borne in mind that this term, like "adaptation" and "physiology" (all of which overlap widely) covers a fantastically great multitude of interwoven biological mechanisms. This complex is so impressive that it has sometimes been confused with the basis of life processes, of which it really forms a superstructure.[4]

[4] In its application to matters of genetic variation, the term homeostasis has recently been given a special meaning [13], whereby it denotes an essentially mystical doctrine, representing a revival from pre-Mendelian times. According to this doctrine, an organism's vigor is *per se* enhanced as a result of the hereditary elements derived from its two parents being unlike one another.

It must not be forgotten that stability is itself only a means of helping to insure survival and reproduction, and that oftentimes these ends are better achieved by lability. In fact, the entire set of physiological reactions of the organism serves as a grand series of examples of what may be called phenotypic lability [11a]. In each normal case, these reactions or changes that the organism undergoes in given situations, instead of representing the passive yielding to environmental pressures that is characteristic of inanimate objects, constitute adaptive responses. By this is meant responses that serve to fend off a danger to survival and/or multiplication, or that serve to take advantage of an opportunity to promote these end-results.

One might beg the question here by declaring that in these cases a deeper or higher stability is served, that of the species or genes themselves. However, this maneuver would stretch the term stability too widely inasmuch as these adaptations are ultimately directed not toward stasis but toward multiplication, of a kind that characteristically takes the form of expansion combined with evolution. It should also be emphasized that these adaptive changes, just like the stabilizing reactions (when, as is only sometimes the case, a distinction can be made!) represent no fundamental property of adaptation on the part of living matter. They are secondary developments, representing (as we have noted in the foregoing discussion) the consequences of the interminably repeated survival and multiplication of the mutant types that happened to be successful.

In addition to the adaptive reactions that have been developed with the function of promoting the welfare of the body proper there are of course those that promote its multiplication. And among the latter are some, of which the prime example is genetic recombination, implemented by sexual reproduction, the function of which is the facilitation of further evolution. Nevertheless, there are no grounds for suspecting that mechanisms have ever become developed that can direct the course of mutation into helpful rather than harmful channels. The process of mutation represents for the organism the taking of an untried step, to the consequences of which it is blind, but which it is ready to profit by if it should make a lucky strike.

It may be reiterated here that the favorable mutation is seldom a large one, and that the smaller its effect is, the more chance there is of its being helpful, and taking part in evolution [10b]. Never do highly organized structures that function helpfully in new ways come into existence at a bound, as they do in "science fiction" stories in which a child is born with telepathic antennae. All organs, tissues

and useful bodily reactions represent the remains of interminable trials, big and little errors that passed away, and little but accumulated successes.

8. **Mutation as destroyer or creator.** Because detrimental mutations are necessarily so much more abundant than favorable ones, it is evident that whenever the rate of elimination of the harmful mutant genes is slackened, as under conditions of easy living or artificial aids, their frequency will tend to rise, and the population will thereby fall off in its natural vigor and in the effectiveness of all reactions of types that have been thus protected [**14**]. For it is, in a sense, only selection that holds the body in shape, like the walls of a vessel containing a gas. Thus in the course of evolutionary time, "mutation pressure" will inevitably take advantage of any yielding of the selectional walls and allow the mass to lose its previous nicely adapted form, just as happens with the creatures who after countless generations in caves are found to be no longer capable, genetically, of forming functional eyes.

Sometimes these retrograde developments are all to the good, as when man, having adopted clothing, became relatively hairless, or when, following the practice of cooking and cutting his food, his great jaws receded. However, this process can be carried too far if society not only does all it can to help its genetically unfortunate members, as it certainly should, but if it also gives them every encouragement and assistance in passing along their weaknesses. By an indefinite continuation of this process, society would become overburdened. Moreover, it is evident that an increase in the pressure of mutations, caused for example by excessive radiation, would exert an influence that worked in the same direction as a relaxation of selection.

I have no fear that the course of mutational deterioration will go to serious extremes, because men are in the process of rapid learning. If they can now avoid self-made disaster they can enter a period of increasing hope and achievement. The rapid recent changes in their ways will cause them to revaluate their ancient standards. They will then see that, by realistically appraising both the world without and the world within themselves, by learning their own basic structure and reactions and the methods of controlling them, they can even challenge and improve upon the results of that greatest of creative operations, biological evolution itself. But first let them open their eyes and become aware of this living world for what it is.

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RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON 2-SPHERES IN 3-MANIFOLDS

BY J. H. C. WHITEHEAD

Communicated by Norman Steenrod, March 7, 1958

1. Let M be a connected, orientable, triangulated 3-manifold and let $\Lambda \subset \pi_2(M)$ be a sub-group which is invariant under the operators in $\pi_1(M)$. By a $\pi_1(M)$-class in $\pi_2(M)$ we mean the set of elements $\pm\xi a$, for some $a \in \pi_2(M)$ and every $\xi \in \pi_1(M)$. The invariance under $\pi_1(M)$ means that, if $a \in \Lambda$, then the entire $\pi_1(M)$-class $\{\pm\xi a\}$ is contained in Λ. A $\pi_1(M)$-class is represented by a map $S^2 \to M$, without reference to base-points or orientation. We shall describe such a map, or singular sphere, as *essential* mod Λ if, and only if, the corresponding $\pi_1(M)$-class is not contained in Λ. The terms polyhedral, piecewise linear etc., when applied to M, will refer to the piecewise affine structure which M derives from the given triangulation K. Thus a polyhedron in M is the carrier of a subcomplex, L, of a (rectilinear) subdivision of K. The polyhedron is compact if, and only if, L is a finite complex.

The main purpose of this note is to show how the proof of the (qualified) sphere theorem, due to C. D. Papakyriakopoulos [5], can be modified so as to yield a proof of:

THEOREM (1.1). *If* $\Lambda \neq \pi_2(M)$, *then M contains a non singular polyhedral 2-sphere which is essential* mod Λ.

On taking $\Lambda = 0$ in (1.1) we obtain the sphere theorem in full generality.

By attaching 3-cells to M we can imbed it in a space X such that Λ is the kernel of the injection $\pi_2(M) \to \pi_2(X)$. Hence it follows that (1.1) is equivalent to:

THEOREM (1.2). *If $M \subset X$, where X is a topological space, and if there is a map $S^2 \to M$, which is essential in X, then M contains a non-singular, polyhedral 2-sphere which is essential in X.*

In particular if X is any orientable 3-manifold, which need not be paracompact, and if f: $S^2 \to X$ is an essential map, then every neighbourhood in X of fS^2 contains a nonsingular 2-sphere, which is

essential in X and polyhedral in some paracompact, and hence triangulable [3], neighbourhood of fS^2.

If M is compact and unbounded, then $\pi_2(M)$ is a free Abelian group whose rank is 0, 1 or ∞ according as $\pi_1(M)$ has less than 2, 2 or ∞ ends [7]. Hence it follows that $\pi_2(M)\neq 0$ if M is a free product with two nontrivial factors. On the other hand if M is orientable, $\pi_2(M)\neq 0$ and $\pi_1(M)$ is not cyclic, then it is easily deduced from (1.1) that M contains an essential, nonsingular (polyhedral) 2-sphere S which separates M. In this case, therefore $\pi_1(M)$ is a nontrivial free product (see a forthcoming paper by J. W. Milnor). I hope to publish, in the Colloquium Mathematicum, a proof that, if $\pi_1(M)$ is a nontrivial free product, then M contains an essential, nonsingular 2-sphere, even if M is nonorientable (cf. [1]).

The group $Z+Z_2$ is neither cyclic nor, being Abelian, a nontrivial free product. Therefore the example $\Lambda=0$, $M=S^1\times P^2$, where P^2 is a real projective plane, shows that (1.1) is false for nonorientable manifolds.

I have been helped in the preparation of this paper by many discussions with J. W. Milnor.

2. **Proof of** (1.2). Let D_0 be a canonical singular 2-sphere in Int (M), essential in X and having the smallest (t, d)-index among all such singular 2-spheres (cf. [5, no. 19]). We build a tower over D_0, using the method and notations of [5] with certain modifications. If $\pi_1(D_0)$ is finite the tower consists of $M=M_0\supset V_0\supset D_0$. If $\pi_1(D_0)$ is infinite let $\tilde{D}_0$ be a universal cover of D_0. Let $f_0\colon G\to D_0$ be as in §10 of [5], where G is a 2-sphere, and let f_0 be lifted to $\tilde{f}_0\colon G\to\tilde{D}_0$. We identify $\pi_1(D_0)$ with the group of covering transformations of $\tilde{D}_0$ in the usual way. Clearly $\tilde{f}_0G=D^*$, say, is a fundamental region in $\tilde{D}_0$ and since D^* is compact, $\tilde{D}_0$ connected and noncompact, there is a $\tau\in\pi_1(D_0)$ such that $\tau\neq 1$, $D^*\cap\tau D^*\neq\phi$. Let (τ) be the sub-group of $\pi_1(D_0)=\pi_1(V_0)$ generated by τ.

We now build the tower as on p. 11 of [5] except that, if $\pi_1(D_0)$ is infinite, then:

(2.1) *M_1 is a cover of V_0 associated with (τ) and M_i is a universal cover of V_{i-1} if $i>1$,*

(2.2) *the construction terminates with the first n such that $\pi_1(V_n)$ is finite.*

Thus the tower is defined for $n\geqq 0$. If $n>0$, then $\pi_1(M_1)\approx(\tau)$, $\pi_1(V_n)$ is finite and the groups $\pi_1(V_0), \cdots, \pi_1(V_{n-1})$ are all infinite. Moreover the projection $\tilde{D}_0\to M_1$ carries $D^*\cap\tau D^*$ into a nonvacuous set of double curves in D_1. Therefore we have:

(2.3) *if $n>0$, then D_1 is singular.*

Let $n \geqq 0$ and suppose that D_n is singular. Then the components of V_n are all spheres because $\pi_1(V_n)$ is finite [6, p. 223]. We refer to No. 22 in [5] with the following modifications. The homotopies in M_0, appearing on pp. 16, 17 of [5], are to be replaced by homotopies in X. Observe that P (see the bottom of p. 16 in [5]) is such that $\pi_1(P) \approx \pi_1(V_n)$, which is finite. Therefore at the top of p. 17 in [5] we have $mK \sim 0$ in P for some $m \geqq 1$ whence the integral intersection number $sc(K, D_n) = 0$ and (22.2) of [5] follows. Moreover $\tilde{P}$, the universal cover of P, has the same homotopy type as S^3 so $\pi_2(P) \approx \pi_2(\tilde{P}) = 0$. Therefore we reach the same contradiction as on p. 17 of [5] and we conclude that D_n is nonsingular.

Since D_n is nonsingular it follows from (2.3) above that $n > 1$ if $n > 0$. In this case $\pi_1(M_{n-1}) \approx (\tau)$ or 0, according as $n = 2$ or $n > 2$. In either case $\pi_1(M_{n-1})$ is Abelian.

Assume that $n > 0$. If $H_1(V_{n-1})$ were infinite, then (23.1) in [5] would follow from [5, (11.2), (12.6)]. The last paragraph in No. 23 of [5] would then lead to a contradiction. Therefore $H_1(V_{n-1})$ is finite. Hence all the components of $\dot{V}_{n-1}$ are spheres and, by (6.1) in [5], the injection $\pi_1(V_{n-1}) \rightarrow \pi_1(M_{n-1})$ is a monomorphism. Since $\pi_1(M_{n-1})$ is Abelian, so is $\pi_1(V_{n-1})$. Therefore $\pi_1(V_{n-1}) \approx H_1(V_{n-1})$ which is finite. This contradicts (2.2). Therefore $n = 0$, D_0 is nonsingular and the proof is complete.

3. **Consequences of** (1.1). Let X be a Hausdorff space and $M \subset X$ a connected 3-manifold such that $M - \dot{M}$ is an open subset of X. Let M be orientable if X is not a 3-manifold (if X is a 3-manifold M need not be orientable).

Theorem (3.1). *The kernel of the injection, $i_*: \pi_1(M) \rightarrow \pi_1(X)$, contains no element of finite order* > 1.

Proof. The group $\pi_1(M) = \pi_1(M, x_0)$, may be identified with $\text{Lim}_{\rightarrow} \{\pi_1(C, x_0)\}$ for every compact $C \subset M$ which contains x_0. Such a C has a paracompact neighbourhood in M and it follows from the triangulation theorem that it is contained in a compact (triangulable) manifold in M. Therefore we may assume that M is compact and triangulated. If M is unbounded it is open and closed in X, whence $i_*: \pi_1(M) \approx \pi_1(X) = \pi_1(X, x_0)$. So we assume that M is bounded.

Let $1 \neq \alpha \in \pi_1(M)$, $\alpha^m = 1$ for some $m > 1$. We have to prove that $i_* \alpha \neq 1$. First let M be nonorientable, X being a manifold. Let X_1 be an orientable cover of X, let $p: X_1 \rightarrow X$ be the projection and let M_1 be the component of $p^{-1}M$ which contains the base point (in $p^{-1}x_0$) for X_1. Then M_1 is orientable. If $i_* \alpha \notin p_* \pi_1(X_1)$, then $i_* \alpha \neq 1$. If $i_* \alpha \in p_* \pi_1(X_1)$, then a loop $I \rightarrow M$, representing α and $i_* \alpha$, lifts into

a loop $I \to M_1$. Therefore $\alpha = p'_* \alpha_1$, where $\alpha_1 \in \pi_1(M_1)$ and $p' = p | M_1$: $M_1 \to M$. Moreover $i_* \alpha = p_* i'_* \alpha_1$, where i': $M_1 \subset X_1$. Since $\alpha \neq 1$ and p_*, p'_* are monomorphisms it follows that $\alpha_1 \neq 1$, $\alpha_1^m = 1$ and that $i'_* \alpha_1 \neq 1$ implies $i_* \alpha \neq 1$. Therefore the theorem will follow when we have proved it for an orientable M and arbitrary X.

If G is any group let $\rho(G)$ be the minimum number of generators among all presentations of G. Thus $0 \leqq \rho(G) \leqq \infty$ and $\rho(G) = 0$ if, and only if, $G = 1$. If $G = G_1 * G_2$, a free product, then $\rho(G) = \rho(G_1) + \rho(G_2)$ [2; 4]. Let $\nu(M)$ be the number of components of $\dot{M}$ and let $\lambda(M) = \rho(\pi_1(M)) + \nu(M)$. Then $\lambda(M) < \infty$ since M is compact. If $\lambda(M) = 0$ there is nothing to prove and we proceed by induction on $\lambda(M)$. It will be enough to sketch the proof because of its similarity to that of (31.2) in [5].

Let us describe the pair (X, M) as *bad* if, and only if, kernel (i_*) contains an element of finite order > 1. Assume that the theorem is false and that, among all bad pairs, (X, M) is one with the smallest $\lambda(M)$. Then M is not aspherical, by (31.1) of [5]. Hence, and since M is bounded, it follows from (25.1) of [5] that $\pi_2(M) \neq 0$. Therefore it follows from (1.1), with $\Lambda = 0$, that M contains a nonsingular, polyhedral 2-sphere S, which is essential in M. We may assume that $S \subset M - \dot{M}$ and, since $\pi_1(M)$ contains an element of finite order > 1, that S separates M. Hence, by cutting[1] through S and filling in the holes, as in [5], we construct a pair (X', M') such that $\lambda(M') < \lambda(M)$ and (X', M') is bad (notice that $\nu(M)$, unlike $n(M)$ in [5], includes the count of the 2-spheres in $\dot{M}$). This contradiction completes the proof.

COROLLARY[2] (3.2). *If X, M are as in* (3.1) *and if $\pi_1(X)$ contains no element of finite order > 1, then $\pi_1(M)$ contains no element of finite order > 1.*

Let A be a connected subset of M which is a compact ANR (for the category of separable metric spaces). Then there is an open subset $U \subset M$ of which A is a retract. Therefore the injection $\pi_1(A) \to \pi_1(U)$ is a monomorphism and from (3.1), applied to X, U, we have:

COROLLARY (3.3). *Let X, M be as in* (3.1) *and let A be a connected, compact ANR in M. Then the kernel of the injection $\pi_1(A) \to \pi_1(X)$ contains no element of finite order > 1.*

---

[1] Since S is a closed subset of X, and because X is a Hausdorff space, and $M - \dot{M}$ is open in X the cutting process can be carried out in the usual way. If X were not a Hausdorff space this would not, in general, be so and (3.1) would be false.

[2] Cf. (31.2) in [5].

Let M, A be as in (3.3), let M be orientable and let $\pi_1(A)$ contain an element, α, of finite order >1. Let f: $S^1 \to A$ be a map which represents α and let g: $S^1 \to M$ be a map such that $f \simeq g$ in M. I say that

$$(3.4) \qquad A \cap gS^1 \neq \emptyset.$$

PROOF. Assume that $A \cap gS^1 = \emptyset$ and let $X = M \cup e^2$, where e^2 is an open 2-cell attached to M by the map g. Since $f \simeq g$ in M it follows that $i_* \alpha = 1$, where i: $A \subset X$. This contradicts (3.3), applied to X, $M - gS^1$, A, and (3.4) is proved.

On considering a cone, X, with a real projective plane, A, as base and $M =$ "X minus vertex" we see that (3.1), $\cdots$, (3.4) are not necessarily true if M is nonorientable. But in (3.2), (3.3), as in (3.1), M may be nonorientable provided X is a 3-manifold.

Let π be any group and G a π-module. By a set of *π-generators* for G we mean a sub-set $B \subset G$ such that every element of G is of the form $\sum_{b \in B} \xi_b b$, where ξ_b is in the integral group ring of π and $\xi_b = 0$ for almost all b.

Let $M = M_1 \cup M_2$, where M_1, M_2 are connected 3-manifolds such that $M_1 \cap M_2 = \dot{M}_1 \cap \dot{M}_2 =$ a 2-sphere.

LEMMA (3.5). *Let B_λ be a set of $\pi_1(M_\lambda)$-generators for $\pi_2(M_\lambda)$ and let ι_λ: $\pi_2(M_\lambda) \to \pi_2(M)$ be the injection* ($\lambda = 1, 2$). Then *$\iota_1 B_1 \cup \iota_2 B_2$ is a set of $\pi_1(M)$-generators for $\pi_2(M)$.*

This follows from the Mayer-Vietoris theorem, applied to a universal cover of M.

Let M be a connected, compact (possibly bounded) orientable 3-manifold.

THEOREM (3.6). *$\pi_1(M)$ has a finite set of $\pi_1(M)$-generators, whose $\pi_1(M)$-classes are represented by disjoint, nonsingular, polyhedral 2-spheres.*

PROOF. The assertion is trivial if $\pi_2(M) = 0$ so we assume that $\pi_2(M) \neq 0$. Then it follows from (1.1) with $\Lambda = 0$, that M contains an essential, nonsingular, polyhedral 2-sphere S. Let $\lambda(M)$ be as in the proof of (3.1). Clearly $\pi_2(M) = 0$ if $\lambda(M) = 0$ or if $\pi_1(M) = 1$ and $\dot{M}$ consists of a single 2-manifold, necessarily a 2-sphere. Therefore, if $\lambda(M) = 1$, then M is closed and $\pi_1(M)$ is cyclic infinite. In this case the manifold obtained from M by cutting through S is 1-connected, and the assertion follows without difficulty from the Hurewicz theorem, applied to a universal cover of M.

If $\lambda(M) > 1$ we may assume that S separates M. Then $M = M_1 \cup M_2$, $M_1 \cap M_2 = S$, where M_1, M_2 are orientable 3-manifolds. Let N_i

$=M_i \cup B_i$ $(i=1,\ 2)$, where B_i is a 3-dimensional ball such that $S=\dot{B}_i=B_i \cap M_i$. Clearly $\lambda(N_i)<\lambda(M)$. Moreover the kernel of the injection $\pi_2(M_i) \rightarrow \pi_2(N_i)$ is generated by the $\pi_1(M_i)$-class represented by S. Also, if $b \in B_i - S$ there is a piecewise linear isotopy of $N_i - b$ into $M_i - S$. Therefore (3.6) follows from induction on $\lambda(M)$ and (3.5).

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ON ISOMORPHISMS OF GROUP ALGEBRAS

BY WALTER RUDIN[1]

Communicated March 7, 1958

With every locally compact topological group G there is associated its group algebra $L(G)$, the space of all complex Haar-integrable functions on G with convolution as multiplication. Considerable work has been done toward discovering the extent to which the algebraic structure of $L(G)$ determines G (see [1; 2; 5]), but some very specific questions have been left unanswered. For instance: Is the group algebra of the circle isomorphic to that of the torus? The theorem announced here stems from this question.

THEOREM. *The group algebra of a locally compact topological group T is isomorphic to that of the circle group C if and only if T is a direct sum $C+F$, where F is a finite abelian group.*

The proof leans heavily on that of Theorem 1 of [4]. In the outline below we will mainly be concerned with pointing out the changes in [4] which are needed to yield the stated result.

If $L(T)$ and $L(C)$ are isomorphic, then T is abelian, and the dual group Γ of T is homeomorphic to J, the group of all integers (the dual group of C) [2, p. 478]. Thus Γ is discrete and countable, and T is a compact abelian group with countable base.

Abelian groups will be written additively; for $x \in T$ and $\phi \in \Gamma$ the symbol (x, ϕ) will stand for the value of the character ϕ at the point x; the Haar measure on T will be denoted by m.

LEMMA 1. *Corresponding to every $E \subset T$ with $m(E) > 0$, there is only a finite set of characters ϕ such that, for all $x \in E$,*

$$|1 - (x, \phi)| < 1. \tag{1}$$

Note that (1) holds if and only if the real part of (x, ϕ) exceeds $1/2$. If f is the characteristic function of E and if ϕ satisfies (1), then $\left|\int_T (x, \phi) f(x) dx\right| > m(E)/2$, and the lemma follows from the Bessel inequality.

LEMMA 2. *Every infinite subset A of Γ contains an infinite subset B, such that for some $x \in T$ the inequality*

$$|1 - (x, \phi)| \geqq 1 \tag{2}$$

holds for every $\phi \in B$.

[1] Research Fellow of the Alfred P. Sloan Foundation.

This is proved by repeated application of Lemma 1.

If now ψ is an isomorphism of $L(T)$ onto $L(C)$, ψ can be extended to an isomorphism of the measure algebras $M(T)$ and $M(C)$, and [2, p. 479] there is a one-to-one mapping α of J onto Γ such that the Fourier-Stieltjes coefficients of $\psi(\mu)$ are

$$c_n(\psi(\mu)) = \int_T (-x, \alpha(n))d\mu(x) \qquad (n \in J, \mu \in M(T)). \tag{3}$$

For $x \in T$, let e_x be the measure of mass 1 which is concentrated at x, and put $\mu_x = \psi(e_x)$. Then $c_n(\mu_x) = (-x, \alpha(n))$, and

$$\mu_x * \mu_y = \mu_{x+y} \qquad (x, y \in T). \tag{4}$$

The mapping $x \rightarrow \mu_x$ is thus an isomorphism of T into $M(C)$.

The discrete parts λ_x of μ_x also satisfy (4), and there is a mapping β of J into Γ such that

$$c_n(\lambda_x) = (-x, \beta(n)) \qquad (n \in J, x \in T); \tag{5}$$

the lemma used in Step 5 of [4] must here be applied to $C \times T$ in place of $C \times C$. Since λ_x is discrete, $c_n(\lambda_x)$ is an almost periodic function on J, for each $x \in T$. Arguing as in Step 6 of [4], we find that there is a positive integer k and a set $E \subset T$ with $m(E) > 0$, such that

$$| 1 - (x, b(n)) | < 1 \qquad (n \in J, x \in E), \tag{6}$$

where $b(n) = \beta(n+k) - \beta(n)$. By Lemma 1, the sequence $\{b(n)\}$ has only a finite set of values, so that the almost periodicity of $\{(x, b(n))\}$ implies that $\{(x, b(n))\}$ is actually periodic, for every $x \in T$. A compactness argument now shows that $\{b(n)\}$ is itself periodic, with period p, say. If $q = kp$, it follows that

$$\beta(n + q) + \beta(n - q) = 2\beta(n) \qquad (n \in J). \tag{7}$$

Next we put $\tau_x = (\lambda_x - \mu_x) * \lambda_{-x}$, so that

$$c_n(\tau_x) = 1 - (x, \gamma(n)) \qquad (n \in J, x \in T), \tag{8}$$

where $\gamma(n) = \beta(n) - \alpha(n)$. Since the measures τ_x are continuous,

$$\lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{-N}^{N} c_n(\tau_x) = 0 \qquad (x \in T). \tag{9}$$

These averages are uniformly bounded on T, so that (9) may be integrated; combined with (8), this implies that $\gamma(n) = 0$ except possibly on a set $S \subset J$ of density 0.

Thus if S is infinite, S contains an infinite set $\{n_k\}$ such that none

of the integers $n_k+1, n_k+2, \cdots, n_k+k$ belong to S, and by Lemma 2 there is an $x \in T$ and a subsequence of $\{n_k\}$, again denoted by $\{n_k\}$, such that $|c_{n_k}(\tau_x)| \geqq 1$. A subsequence of the measures

$$d\sigma_k(\theta) = e^{-in_k\theta} d\tau_x(\theta) \tag{10}$$

then converges weakly to a singular measure σ [3, p. 236] with $|c_0(\sigma)| \geqq 1$ but $c_n(\sigma) = 0$ for all $n > 0$. This is impossible, so that S is finite.

It follows that $\alpha = \pi\beta$, where β satisfies (7) and maps J onto Γ, and π is a permutation of Γ which moves only a finite number of terms; β maps each residue class mod q onto an arithmetic progression in Γ; hence Γ is finitely generated and is therefore a direct sum of a finite set of cyclic groups; since Γ is the union of a finite set of arithmetic progressions, only one of the direct summands can be infinite, so that Γ is a direct sum of J and a finite abelian group F.

This proves one half of the theorem. The converse may be proved by defining

$$\alpha(nq + k) = (n, f_k) \qquad (n \in J,\ 1 \leqq k \leqq q), \tag{11}$$

where $f_1, \cdots, f_q$ are the elements of F; it is easily verified that this induces, via (3), an isomorphism of $L(T)$ onto $L(C)$. In fact, every α of the above form $\alpha = \pi\beta$ has this property, as can be seen by an argument analogous to that on p. 50 of [4].

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A CLASS OF LATTICE ORDERED ALGEBRAS[1]

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Communicated by Garrett Birkhoff, April 1, 1958

1. Our purpose is to characterize those lattice ordered algebras which may be represented as algebras of Carathéodory functions. This work is, accordingly, a sequel to [1] where the same problem was considered for lattice ordered groups. The rings considered here are more restrictive than those of Birkhoff and Pierce in [2], where an "F-ring" is shown to be isomorphic to a subring of the direct union of totally ordered rings (but the multiplication in [2] is not necessarily that which may be expected for functions; indeed, all products may be zero. In our case, the axioms compel the algebra multiplication to conform to that of the Carathéodory functions). Brainerd [3] has considered a class of algebras which have function space representations, but his emphasis is different from ours.

2. In this section, we define a Carathéodory algebra. Let B be a relatively complemented distributive lattice. Let E be the set of forms $f=a_1\alpha_1+\cdots+a_n\alpha_n$, where $\alpha_i\in B$, a_i real, $i=1,\cdots,n$. With $f\geqq 0$ if $a_i\geqq 0$ for all i, and addition and multiplication defined by $f+g=\sum_{i=1}^{n}\sum_{j=1}^{m}(a_i+b_j)(\alpha_i\cap\beta_j)+\sum_{i=1}^{n}a_i(\alpha_i-\bigcup_{j=1}^{m}\beta_j)+\sum_{j=1}^{m}b_j(\beta_j-\bigcup_{i=1}^{n}\alpha_i)$ and $fg=\sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j(\alpha_i\cap\beta_j)$ where $f=\sum_{i=1}^{n}a_i\alpha_i$ and $g=\sum_{j=1}^{m}b_j\beta_j$, E is a lattice ordered algebra, which we call the algebra of elementary Carathéodory functions. Let $\overline{E}$ be the conditional completion of E. $\overline{E}$ is the set of bounded Carathéodory functions. In order to define the general Carathéodory function, we need the notion of carrier. In a lattice ordered group, for every $x\geqq 0$, $y\geqq 0$, we say $x\sim y$ if $x\cap z=0$ when and only when $y\cap z=0$. The equivalence classes obtained in this way are called carriers (filets by Jaffard [4]) and form a relatively complemented distributive lattice. The equivalence class to which x belongs is called the carrier of x. In $\overline{E}$, consider pairwise disjoint sequences $\{f_n\}$ whose carriers have an upper bound, and consider the formal sums $\sum f_n$. With order, addition, and multiplication defined appropriately, these formal sums constitute a lattice ordered algebra—the Carathéodory algebra C generated by B. (For details on related matters see [5; 6] and [1].)

3. Let R be an archimedean lattice ordered algebra. Then R is a lattice with positive cone P such that x, $y\in P$, $a\geqq 0$ real, implies

[1] Supported by National Science Foundation grant no. NSF G-2267 on ordered systems.

$x+y$, xy, $ax \in P$, and if x, $y \in P$, $y>0$, implies there is a real $a \geqq 0$ with $x-ay \notin P$. We say that R is totally complete if

(a) R is conditionally complete.

(b) every sequence of pair-wise disjoint elements in P, whose sequence of carriers has an upper bound, itself has an upper bound; hence, a least upper bound.

In addition to the archimedean hypothesis, the following condition is important for us.

A. If x, y, z are in P(i.e., $x \geqq 0$, $y \geqq 0$, $z \geqq 0$) then $(xy) \cap z=0$ if and only if $x \cap y \cap z=0$.

It is not hard to see that the Carathéodory algebra C is totally complete and satisfies A.

4. Before considering the main problem, we point out that for every totally complete vector lattice R, multiplication may be defined so that R is an algebra satisfying A. We outline the procedure.

Let $[u_\alpha]$ be a generalized weak unit [1] in R. Then, for every carrier α, there is a unique u_α with carrier α, and for every α, β we have $u_\alpha \cap u_\beta = u_{\alpha \cap \beta}$ and $u_\alpha \cup u_\beta = u_{\alpha \cup \beta}$. For every $x>0$ there is, by the total completeness of R, a pairwise disjoint sequence $\{u_{\alpha_n}\}$ and a sequence $\{a_n\}$ of positive reals, such that sup $a_n u_{\alpha_n} \geqq x$. For every $x>0$, $y>0$ let u_{α_n}, a_n be as above relative to x and v_{β_n}, b_n as above relative to y. Let $\xi = \sup\ (a_n u_{\alpha_n})(b_m v_{\beta_m})$. Then define $xy = \inf\ \xi$ for all ξ obtained in this way. For any x, $y \in R$, define $xy = x^+y^+ + x^-y^- - x^+y^- - x^-y^+$. It can then be shown that R is an algebra satisfying A. Moreover, if R has a weak unit, the resulting algebra has an identity.

5. We now let R be a totally complete lattice ordered algebra, satisfying A.

LEMMA 1. *If* $x \geqq 0$, $y \geqq 0$ *then* $xy=0$ *if and only if* $x \cap y=0$.

LEMMA 2. *If* $x \geqq 0$ *then* x *and* x^2 *have the same carrier.*

PROOF. $x \cap y=0$ *implies* $x \cap x \cap y=0$ implies $x^2 \cap y=0$. Conversely, $x^2 \cap y=0$ implies $x \cap x \cap y=0$ implies $x \cap y=0$. More generally,

LEMMA 2′. *If* x, $y \geqq 0$ *have the same carrier, then* xy *also has this carrier.*

COROLLARY 1. *Every carrier is a semi-ring.*

Since R is conditionally complete, for every x, $y \in R$, the projection y_x of x on y is defined.

LEMMA 3. $xy = xy_x$.

The next lemma is important for us.

LEMMA 4. *If* $x>0$ *there is* $y>0$ *with* $yx\geqq x$ *and* $z>0$ *with* $zx\leqq x$.

We outline the proof. From Lemma 2, the supremum of the carriers α_n of $w_n=(nx^2-x)^+$ is the carrier of x. Let $\beta_n=\alpha_n-\alpha_{n-1}$ and let z_n have carrier β_n. If $y_n=(nx)_{z_n}$, the y_n are pair-wise disjoint. By the total completeness of R, sup $y_n=y$ exists. Then $yx\geqq x$. The proof of the second part is similar.

DEFINITION. For every $x\geqq 0$, $u(x)=\inf\ [y|\, yx\geqq x]$ and $\bar{u}(x)=\sup\ [y|\, yx\leqq x]$.

LEMMA 5. *For every* $x\geqq 0$, $x=u(x)x=\bar{u}(x)x$.

PROOF. $u(x)x\geqq x$. If $u(x)x>x$ there is $z>0$ with $zx<u(x)x-x$, whereby $(u(x)-z)x>x$, which is impossible.

LEMMA 6. $[u(x)]^2=u(x)$ *and* $[\bar{u}(x)]^2=\bar{u}(x)$.

PROOF. $[u(x)]^2x=u(x)[u(x)x]=u(x)x=x$ so that $[u(x)]^2\geqq u(x)$. Similarly, $[\bar{u}(x)]^2\leqq\bar{u}(x)$. But $\bar{u}(x)x=x$ implies $\bar{u}(x)\geqq u(x)$. However, $\bar{u}(x)\leqq u(x)$.

COROLLARY 2. $u(x)=\bar{u}(x)$.

LEMMA 7. *The carriers of* x *and* $u(x)$ *are the same.*

PROOF. By condition A.

LEMMA 8. *If* x *and* y *have the same carrier then* $u(x)=u(y)$.

PROOF. If $0<x<z<y$ and $x^2=x$, $y^2=y$ then $z^2=z$. Let α be the carrier of x and y. If $u(x)\neq u(y)$, there is $\beta<\alpha$ and $k<1$ such that, say, $k(u(x))_w<(u(y))_w$, where w has β as carrier. But then $[k(u(x))_w]^2=k(u(x))_w$ and $k(u(x))_w=(u(x))_w$. This is impossible.

Thus there is a one-one correspondence $\alpha\rightarrow u_\alpha$ between the carriers and idempotents. There is a unique left identity for every carrier relative to the carrier; there is also a unique right identity.

LEMMA 9. *For every* α, *the associated right and left identities are equal.*

PROOF. Both are idempotents. The proof is then as for Lemma 8. We summarize:

THEOREM 1. *A totally complete lattice ordered algebra* R *satisfying* A *has a unique idempotent* u_α *with carrier* α, *for every* α. *The idempotent* u_α *is an identity* (*left and right*) *for all* $x\in R$ *whose carrier is* $\leqq\alpha$.

COROLLARY 3. *The family* $[u_\alpha]$ *is a generalized weak unit in* R.

Proceeding as in [1], the algebra R can be reconstructed from the u_α and a one-one correspondence obtained between the elements of R and those of the space C of Carathéodory functions generated by the relatively complemented distributive lattice B of carriers in R. In this correspondence, each element $a_1u_{\alpha_1}+ \cdots +a_nu_{\alpha_n}\in R$ is mated with the element $a_1\alpha_1+ \cdots +a_n\alpha_n\in C$. It is then a routine matter to check that this correspondence preserves order, addition, and multiplication. We thus have:

THEOREM 2. *A lattice ordered algebra is isomorphic with the algebra C of Carathéodory functions generated by a relatively complemented distributive lattice if and only if it is totally complete and satisfies A; i.e., for $x, y, z\geqq 0$, $(xy)\cap z=0$ if and only if $x\cap y\cap z=0$.*

The following conditions are closely related to A.

A′. If $x, y\geqq 0$, then $xy=0$ if and only if $x\cap y=0$.

A″. R is an F-ring with no nonzero nilpotents.

Indeed, M. Henriksen has shown (oral communication) that conditions A, A', A'' are equivalent. Using this fact, and a completion theorem of Nakano [7] we obtain:

COROLLARY 4. *An archimedean lattice ordered algebra which satisfies A, and is such that* $\inf S=0$ *and* $x\geqq 0$ *implies* $\inf xS=0$, *is isomorphic with a subalgebra of a Carathéodory algebra.*

We also obtain the following fact, which was proved in a different way for F-rings by Birkhoff and Pierce.

COROLLARY 5. *An archimedean lattice ordered algebra which satisfies A has commutative multiplication.*

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PURDUE UNIVERSITY AND
UNIVERSITY OF OKLAHOMA

A PROOF AND EXTENSION OF DEHN'S LEMMA

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Communicated by Norman Steenrod, April 25, 1958

1. **Introduction.** C. D. Papakyriakopoulos [2] has recently proved Dehn's lemma [1]. His proof has the merit that the basic construction (the tower) and the crucial lemmas apply to the sphere theorem as well as Dehn's lemma. However, if one is content with Dehn's lemma the proof can be simplified. In this note we first give a simplified proof of Dehn's lemma and then prove an analogous theorem for surfaces with more than one boundary curve.

By a *surface of type* (p, r) we mean a connected, compact, orientable surface of genus p with a boundary consisting of r 1-spheres. Thus the Euler characteristic of such a surface is $2(1-p)-r$. A surface of type $(0, 1)$ is called a *disc*. A *Dehn surface* of type (p, r) is a polyhedral singular surface of type (p, r) with no singularities on the boundary. Our extension of Dehn's lemma refers to surfaces of type $(0, r)$. In order to state it we need two more definitions.

Let M be a connected 3-manifold and let $\tilde{M}$ be a universal cover of M. The manifold $\tilde{M}$ is orientable and the set of elements of $\pi_1(M) = \pi_1(M, x_0)$ which correspond, in the usual way, to orientation-preserving covering transformations of $\tilde{M}$ is a sub-group of index 2. We denote it by $\omega(M)$. Thus M is orientable if, and only if, $\omega(M) = \pi_1(M)$.

Let $C_1, \cdots, C_r$ be nonsingular closed curves in M and let each C_r be oriented and joined to x_0 by a path in M so as to represent an element $\alpha_i \in \pi_1(M)$. The smallest invariant sub-group of $\pi_1(M)$ which contains $\alpha_1, \cdots, \alpha_r$ is independent of the orientations of $C_1, \cdots, C_r$ and of the paths joining them to x_0. We denote it by $\{C_1, \cdots, C_r\}_M$, or by $\{C_1, \cdots, C_r\}$ when there is no danger of confusion.

We can now state our theorem, with the usual qualifications concerning piecewise linearity (see §2 below).

THEOREM (1.1). *Let $(C_1, \cdots, C_r)$ be a set of disjoint, nonsingular 1-spheres in M which constitute the boundary of a Dehn surface of type $(0, r)$. Let $\{C_1, \cdots, C_r\} \subset \omega(M)$. Then a nonvacuous sub-set of $(C_1, \cdots, C_r)$, say $(C_1, \cdots, C_q)(0 < q \leqq r)$, constitute the boundary of a nonsingular surface of type $(0, q)$.*

If $r = 1$, then $\{C_1\} = 1 \in \omega(M)$ and (1.1) reduces to Dehn's lemma.

2. **Preliminaries.** It is to be understood that every n-manifold $(n = 1, 2, 3)$ to which we refer has a definite piecewise affine structure

and that all our maps are piecewise linear. Likewise a singular (or nonsingular) curve or surface in a 3-manifold will mean the image of some standard polyhedron in a piecewise linear map. As in [2] a singular surface in a 3-manifold M will be called *normal* if it has no singularities other than double lines, triple points and branch points. It will be called *canonical* if it is normal and has no branch points.

By a *Dehn set* of curves in M we shall mean a finite, nonvacuous set of nonsingular, disjoint (polyhedral) 1-spheres whose union is the boundary of a Dehn surface of genus 0. Such a set will be called *good* (*bad*) if it contains (does not contain) a nonvacuous sub-set which constitutes the boundary of a nonsingular Dehn surface of genus 0. Thus (1.1) states that the set $(C_1, \cdots, C_r)$ is good provided $\{C_1, \cdots, C_r\} \subset \omega(M)$. If $r=1$, then C_1 is called a *Dehn curve.* Any nonsingular (polyhedral) closed curve in $\dot{M}$ is obviously a Dehn curve if it is inessential in M.

Let D be a Dehn surface in M. According to our conventions there is a (rectilinear) triangulation, K, of M with a sub-complex, L, which covers D. Let K'', L'' be the second derived complexes of K, L and let V be the union of all the (closed) simplexes of K'' which contain vertices of L'' in $D-\dot{D}$. Then V is a bounded 3-manifold, $\dot{D}\subset\dot{V}$ and D is a deformation retract of V.

Let $\alpha\in\pi_1(V)$. Then $\alpha\in\omega(V)$ if, and only if, the transport of an indicatrix round a loop representing α preserves local orientation. Hence it follows that $\{C_1, \cdots, C_r\}_V\subset\omega(V)$ if, and only if, $\{C_1, \cdots, C_r\}_M\subset\omega(M)$. Therefore we may replace M by V when proving (1.1).

3. **Proof of Dehn's lemma.** Let C be a Dehn curve on the boundary of a compact, connected 3-manifold V.

LEMMA (3.1). *If V has no 2-sheeted cover, then C is good.*

PROOF. Assume that V has no 2-sheeted cover. Then it is orientable. If $H_1(V)$ has infinite order, then, since $H_1(V)$ is finitely generated, it has a cyclic infinite direct summand. There is, therefore, an epimorphism $H_1(V)\rightarrow Z_2$. On composing this with the Hurewicz homomorphism $\pi_1(V)\rightarrow H_1(V)$ we have an epimorphism ϕ: $\pi_1(V)\rightarrow Z_2$. The kernel of ϕ has index 2 and determines a 2-sheeted cover of V, contrary to our assumption. Therefore $H_1(V)$ is finite and, since V is orientable, every component of $\dot{V}$ is a 2-sphere [3, p. 223]. Thus C lies in a nonsingular 2-sphere in $\dot{V}$ and (3.1) follows.

Let C, V be as above and now assume that V has a 2-sheeted cover V_1. Let p: $V_1\rightarrow V$ be the projection and τ: $V_1\rightarrow V_1$ the covering trans-

formation other than the identity. Then $p^{-1}C=C_1\cup\tau C_1$, where C_1 is a Dehn curve in $\dot{V}_1$, and $C_1\cap\tau C_1=\phi$.

LEMMA (3.2). *If C_1 is good, so is C.*

PROOF. Let $C_1=\dot{D}_1$, where D_1 is a nonsingular disc. We assume, as we obviously may, that $D_1\cap\dot{V}_1=C_1$, whence $C_1\cap\tau D_1=\tau C_1\cap D_1=\phi$. Then $pD_1=D$, say, is a Dehn disc bounded by C. Clearly D may be normalized without introducing singularities in D_1 (each step in the normalization is a "local isotopy"). So we assume that D is normal, and hence canonical because p is a local homeomorphism. Then $D_1\cap\tau D_1$ consists of disjoint, nonsingular closed curves and D has no triple points. Since there are no triple points the lemma follows from Dehn's original argument [1; see also 2, no. 14].

Now let C be a given Dehn curve in M and D a canonical Dehn disc bounded by C. Let $d(D)$ denote the number of closed (possibly self-interesting) double curves in D. Let V be as in §2. Thus $D\subset V\subset M$, $C\subset\dot{V}$ and D is a deformation retract of V. Let V have a double cover V_1 and let p: $V_1\rightarrow V$, τ: $V_1\rightarrow V_1$ be as above. Let D be canonical with respect to a map f: $\Delta\rightarrow V$, where Δ is a 2-simplex ($D=f\Delta$), and let f_1: $\Delta\rightarrow V_1$ be such that $pf_1=f$. Then $f_1\Delta=D_1$, say, is a canonical Dehn disc in V_1 and $p^{-1}D=D_1\cup\tau D_1$. Since $D_1\cap\tau D_1\neq\phi$, because D is a deformation retract of V, it follows without difficulty (cf. (9.1) in [2]) that $d(D_1)<d(D)$. Therefore Dehn's lemma follows from (3.1), induction on $d(D)$ and (3.2).

4. **Proof of** (1.1). Let $C_1, \cdots, C_r$ $(r>0)$ be a set of disjoint, nonsingular 1-spheres on the boundary of a connected, compact 3-manifold V. Let C_i also denote a basic 1-cycle carried by the 1-sphere C_i and let $m_1C_1+\cdots+m_rC_r=\partial D$, where $m_i>0$ and D is a singular 2-chain in V. This condition is obviously satisfied if $(C_1, \cdots, C_r)$ is a Dehn set. Let $\{C_1, \cdots, C_r\}\subset\omega(V)$.

LEMMA 4.1. *If V has no 2-sheeted covering, p: $V_1\rightarrow V$, such that $\{C_1, \cdots, C_r)\subset p_*\pi_1(V_1)$, then some nonvacuous sub-set of $(C_1, \cdots, C_r)$ is a good Dehn set.*

PROOF. Assume that there is no such covering. Then V is orientable because $\{C_1, \cdots, C_r\}\subset\omega(V)$. Let G be the image of the injection $H_1(C_1\cup\cdots\cup C_r)\rightarrow H_1(V)$. The Hurewicz homomorphism $\pi_1(V)\rightarrow H_1(V)$ obviously carries $\{C_1, \cdots, C_r\}$ into G. Therefore it follows from the first part of the proof of (3.1), with $H_1(V)$ replaced by $H_1(V)/G$, that $H_1(V)/G$ is a finite group. Therefore every 1-cycle in V is homologous, with rational coefficients, to a linear combination

of $C_1, \cdots, C_r$. The algebraic intersection[1] $C_i \cdot D$ is 0 since V and hence $\dot{V}$ are orientable. Therefore it follows that there is no 1-cycle C_0 in $\dot{V}$ such that $C_0 \cdot C_j = 1$, $C_0 \cdot C_k = 0$ if $j \neq k$. Also $p^1(V) \leqq r$, where $p^1(V)$ is the first Betti number of V. If $g(\dot{V})$ is the sum of the genera of all the components of $\dot{V}$, then $g(\dot{V}) \leqq p^1(V)$ [3, p. 223]. Therefore $g(\dot{V}) \leqq r$.

Let $U_1, \cdots, U_n$ be the components of $\dot{V} - (C_1 \cup \cdots \cup C_r)$ and let $S_\lambda = \overline{U}_\lambda$. Then S_λ is a nonsingular surface of type (p_λ, r_λ), say, and $\dot{S}_\lambda \subset C_1 \cup \cdots \cup C_r$. If $C_j \subset \operatorname{Int}(S_\lambda)$, then nearby points on opposite sides of C_j are joined by a path in U_λ. There is therefore an oriented closed curve $C_0 \subset S_\lambda$ such that $C_0 \cdot C_j = 1$, $C_0 \cdot C_k = 0$ if $k \neq j$, which contradicts a previous conclusion. Therefore each C_i is on the boundary of two of $S_1, \cdots, S_n$ and it follows that $r_1 + \cdots + r_n = 2r$. Hence, if $\chi(X)$ denotes the Euler characteristic of a given space X and if $\dot{V}$ has m components, then

$$2(m - g(\dot{V})) > \chi(\dot{V}) = \sum_\lambda \chi(S_\lambda) = \sum_\lambda (2(1 - p_\lambda) - r_\lambda)$$

and $g(\dot{V}) = m + (p_1 - 1) + \cdots + (p_n - 1) + r$.

If $p_\lambda = 0$, $r_\lambda = 0$, then S_λ is a nonsingular surface of type $(0, r_\lambda)$ bounded by a sub-set of $(C_1, \cdots, C_r)$. So we assume that $p_\lambda > 0$ if $r_\lambda > 0$. Then $r_\lambda = 0$ if $p_\lambda - 1 < 0$. In this case S_λ is a component of $\dot{V}$ which does not contain any of $C_1, \cdots, C_r$. Since $C_i \subset \dot{V}$ it follows that $m + (p_1 - 1) + \cdots + (p_n - 1) > 0$, whence $g(\dot{V}) > r$. This contradicts a previous conclusion and (4.1) is proved.

Let $(C_1, \cdots, C_r)$ be a Dehn set on $\dot{V}$ and let p: $V_1 \to V$ be a 2-sheeted covering of V such that $\{C_1, \cdots, C_r\} \subset p_* \pi_1(V_1)$. Let S be a nonsingular surface of type $(0, r)$ and f: $S \to V$ a map whose image is a Dehn surface bounded by $C_1 \cup \cdots \cup C_r$. Clearly

$$f_* \pi_1(S) \subset \{C_1, \cdots, C_r\} \subset p_* \pi_1(V_1).$$

Therefore f can be lifted to a map f_1: $S \to V_1$ such that $pf_1 = f$. Then $f_1 S$ is a Dehn surface of type $(0, r)$ and $f_1 \dot{S}$ is the union of a Dehn set of curves $(C_1', \cdots, C_r')$ such that $C_i = pC_i'$. Moreover $p^{-1} C_i = C_i' \cup \tau C_i'$, $C_i' \cap \tau C_i' = \phi$, where τ: $V_1 \to V_1$ is the covering transformation other than the identity.

LEMMA (4.2). *If the set* $(C_1', \cdots, C_r')$ *is good, so is* $(C_1, \cdots, C_r)$.

[1] This refers to the intersection pairing $H_1(V) \times H_2(V, \dot{V}) \to Z$. If $i_*: H_q(\dot{V}) \to H_q(V)$ is the injection and ∂: $H_2(V, \dot{V}) \to H_1(\dot{V})$ the boundary homomorphism, then, $i_* a \cdot b = \pm i_*(a \cdot \partial b)$. (Cf. [4, pp. 171, 172].)

The proofs of (4.2) and then of (1.1) are essentially the same as those of (3.2) and Dehn's lemma. The details are left to the reader.

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RESEARCH PROBLEMS

12. Richard Bellman: *Ordinary differential equations.*

It is known that if

a. A is a stability matrix, i.e., all characteristic roots have negative real parts,

b. $\|g(x)\|/\|x\|\to 0$ as $\|x\|\to 0$, $(\|x\| = \sum_i |x_i|)$,

then all solutions of $dx/dt = Ax+g(x)$ approach zero as $t\to\infty$, provided that $\|x(0)\|$ is sufficiently small (Poincaré-Lyapunov theorem).

If $x(0)=a_1c$, where c is a characteristic vector of A and a_1 is a scalar, what is the precise bound for $|a_1|$ in terms of A and $g(x)$? (Received January 7, 1958.)

13. Richard Bellman: *Partial differential equations.*

It is known that if $|g(u)|/|u|\to 0$ as $u\to 0$, then the solution of $u_t=u_{xx}+g(u)$, $u(0, t)=u(1, t)=0$, $t>0$, approaches zero as $t\to\infty$, provided that $\mathrm{Max}_{0\le x\le 1}|u(x, 0)|$ is sufficiently small.

a. If $u(x, 0)=c_1$ what is the precise bound for $|c_1|$ in terms of $g(u)$?

b. If $u(x, 0)=c_1 \sin k\pi x$, what is the precise bound for $|c_1|$ in terms of $g(u)$?

14. Richard Bellman: *Functional equations.*

Let $f_n(u)$ be an analytic function of the function $u(x)$ and its first n derivatives $u'(x), \cdots, u^{(n)}(x)$, for $u\neq 0$, satisfying the functional equation

$$f_n(uv) = f_n(u) + f_n(v).$$

It is well-known that $f_0(u)=c_1 \log u$, and under much lighter conditions, and it is easy to show that $f_1(u)=c_1 \log u+c_2u'/u$.

What is the analytic form of f_n for general n? (Received January 9, 1958.)

15. Richard Bellman: *Functional equations and differential equations.*

Consider the nth order linear differential equation

$$\frac{d^nu}{dt^n} + a_1(t)\frac{d^{n-1}u}{dt^{n-1}} + \cdots + a_n(t)u = 0$$

and make the change of variable $u=vw$. The function w satisfies an equation of the same type with coefficients $b_i(t)$, $i=1, 2, \cdots, n$, where

$$b_1(t) = a_1(t) + nv'/v,$$

$$b_2(t) = a_2(t) + a_1(t)(n-1)v'/v + \frac{n(n-1)}{2} v''/v,$$

and so on.

Introduce the function

$$b_k(t) = f_k(a, v) = f_k(a_1, a_2, \cdots, a_k; v),$$

dependent upon v and its first k derivatives, for $k=1, 2, \cdots, n$. It is easy to see, from the origin of the coefficients b_k, that f_k satisfies the group property

$$f_k(a, v_1v_2) = f_k(f_1(a, v_1), f_2(a, v_1), \cdots, f_k(a, v_1); v_2).$$

What are the most general functions satisfying these functional relations? (Received January 9, 1958.)

16. Richard Bellman: *Functional equations and differential equations.*

If in the above linear equation, we introduce a change in the independent variable of the form $t=\phi(s)$, we obtain coefficients $b_i(t)$ which are functions of the a_i and the derivatives of ϕ. Similarly, the iterated substitution $t=\psi(\phi(s))$ gives rise to functional equations of the type given above. What are the most general functions satisfying these relations? (Received January 9, 1958)

17. Richard Bellman: *Matrix functional equations and differential equations.*

In the matrix differential equation $dX/dt=A(t)X(t)$ make the change of variable $X(t)=Y(t)Z(t)$. Then Z satisfies the equation

$$dZ/dt = (Y^{-1}A(t)Y - Y^{-1}Y')Z.$$

Introduce the matrix function

$$F(A; Y) = Y^{-1}AY - Y^{-1}Y'.$$

Then, as before,

$$F(A; Y_1Y_2) = F(F(A; Y_1), Y_2).$$

What is the most general matrix function of Y and Y' satisfying this equation? (Received January 9, 1958.)

THE APRIL MEETING IN CHICAGO

The five hundred forty-fourth meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 18 and 19, 1958. There was a total of 245 registrations. Among these were 219 members of the Society.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited Professor George Whaples of Indiana University to address the Society. Professor Whaples spoke on the topic *Quasi Galois fields and local class field theory*. Professor Saunders MacLane presided at the session at 2:00 P.M. on Friday, April 18.

The ladies of the Department of Mathematics entertained the Society at a tea on Friday afternoon.

Sessions began at 10:00 A.M. on Friday and the Meeting concluded with a special session for late papers at 1:00 P.M. on Saturday. In all there were eight sessions for the presentation of contributed papers. Presiding Officers were Professors E. J. Mickle, M. F. Smiley, Morris Marden, S. T. Hu, Dr. Stephen Smale, Dr. D. A. Flanders, Professor L. M. Kelly, and Dr. R. S. Palais.

J. W. T. YOUNGS,
Associate Secretary

THE APRIL MEETING IN STANFORD

The five hundred forty-fifth meeting of the American Mathematical Society was held at Stanford University, California, (and at the Stanford Research Institute in Menlo Park) on Friday and Saturday, April 18 and 19, 1958. There were 227 registrants, including 200 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor W. W. Rogosinski of the University of Colorado and the University of Durham addressed the Society on Saturday, his topic being *General moment problems.* He was introduced by Professor Arthur Erdelyi.

By invitation of the same Committee, a Symposium on Banach Algebras and Harmonic Analysis was held on Friday. The program committee for the Symposium consisted of Professors Edwin Hewitt (Chairman), University of Washington, I. I. Hirschman, Jr., Washington University (St. Louis), Henry Helson, University of California (Berkeley), and H. A. Dye, University of Southern California. For the first session of the Symposium, Professor Hewitt presided and hour talks were given by Professor Arne Beurling, Institute for Advanced Study, on *Dirichlet spaces*, and Professor Richard Arens, University of California (Los Angeles) on *Some topics in the theory of commutative Banach algebras.*

The second session of the Symposium was presided over by Professor Helson, the half-hour talks being as follows: *Local operators on Fourier transforms*, by Professor Harry Pollard, Cornell University; *The spectra of multiplier transformations*, by Professor I. I. Hirschman, Jr., Washington University (St. Louis); *Walsh functions and allied topics*, by Professor N. J. Fine, University of Pennsylvania; *Almost periodic functions on semigroups*, by Professor Karel deLeeuw, Stanford University; *Idempotent measures on compact abelian groups*, by Professor Walter Rudin, University of Rochester; *The dual spaces of the complex unimodular groups*, by Professor J. M. G. Fell, University of Washington.

Chairman for the third session was Professor H. A. Dye, the following half-hour talks being presented: *Involutions on Banach algebras*, by Professor Paul Civin, University of Oregon; *On some problems of Gel'fand*, by Dr. Kenneth Hoffman, Massachusetts Institute of Technology; *Function algebras*, by Professor H. L. Royden, Stanford University.

The sessions for contributed papers were presided over by Dr. Joel

Brenner and Professors J. H. Barrett, David Gale, R. P. Dilworth, and W. G. Bade. Mr. Poulsen was introduced by Professor Frantisek Wolf and Dr. Hano by Professor Klee.

VICTOR KLEE,
Associate Secretary

THE APRIL MEETING IN NEW YORK

The five hundred forty-sixth meeting of the American Mathematical Society was held on Thursday, Friday, and Saturday, April 24–26, 1958, at Columbia University. A Symposium on Combinatorial Designs and Analysis (sponsored by the Society with the aid of the Office of Ordnance Research) was held in conjunction with the regular meeting. About 450 persons attended, including 390 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor I. M. Singer of the Massachusetts Institute of Technology addressed the Society on *Connections and holonomy groups* at 2:00 P.M. on Friday, and Professor J. C. Moore of Princeton University delivered an hour address entitled *A survey of some modern developments in homotopy theory* at 2:00 P.M. on Saturday. Professors W. S. Massey and N. E. Steenrod presided at these sessions respectively.

The Symposium was divided into four sessions which met at 10:00 A.M. and 2:00 P.M. on Thursday, 10:00 A.M. on Friday, and 10:00 A.M. on Saturday. At the first session, devoted to Existence and Construction of Combinatorial Designs, participants in the Symposium were welcomed by Lt. Colonel J. B. Sestito of the Office of Ordnance Research, U. S. Army. The following papers were presented: *Current studies on combinatorial designs* by Professor Marshall Hall, Jr. of Ohio State University; *Quadratic extensions of cyclic planes* by Professor R. H. Bruck of the University of Wisconsin; *Homomorphisms of projective planes* by Professor D. R. Hughes of Ohio State University; *The cyclotomic number of order* 10 by Professor A. L. Whiteman of the University of Southern California; *Finite division algebras and finite planes* by Professor A. A. Albert of the University of Chicago.

At the second session, devoted to Combinatorial Analysis of Discrete Extremal Problems, the following papers were presented: *Some recent applications of the theory of linear inequalities to extremal combinatorial analysis* by Dr. A. J. Hoffman of the General Electric Company; *Compound and induced matrices in combinatorial analysis* by Professor H. J. Ryser of Ohio State University; *Duality structure* by Professor A. W. Tucker of Princeton University; *Linear inequalities and the Pauli principle* by Professor H. W. Kuhn of Bryn Mawr College; *On some communication network problems* by Dr. R. E. Kalaba of the RAND Corporation; *Permanents of doubly stochastic matrices* by Dr. Morris Newman of the National Bureau of Standards.

The third session of the Symposium was concerned with Problems of Communications, Transportation and Logistics. The following papers were given: *Dynamic programming and combinatorial processes* by Dr. R. E. Bellman of the RAND Corporation; *A problem in binary encoding* by Dr. E. N. Gilbert of the Bell Telephone Laboratories; *Directed graphs and assembly schedules* by Dr. J. Foulkes of the Bell Telephone Laboratories (in Dr. Foulkes' absence, the paper was read by Dr. V. A. Vyssotsky); *Solutions of large scale transportation problems* by Professor Murray Gerstenhaber of the University of Pennsylvania and the Institute for Advanced Study; *An algorithm for integral solutions of linear programs* by Dr. R. E. Gomory of Princeton University. Discussion at this session was led by Dr. M. M. Flood.

At the final session of the Symposium, devoted to Numerical Analysis of Discrete Problems, the following papers were presented: *Teaching combinatorial tricks to a computer* by Professor D. H. Lehmer of the University of California at Berkeley; *The computational size of the ten by ten orthogonal Latin square problem* by Professors C. B. Tompkins and L. J. Paige of the University of California at Los Angeles; *Some discrete variable computations* by Professor John Todd and Dr. Olga Taussky of the California Institute of Technology: *An enumeration technique for a class of combinatorial problems* by Professor R. J. Walker of Cornell University; *Some combinatorial problems associated with finite partially ordered sets* by Professor R. P. Dilworth of the California Institute of Technology; *Two methods of search in an n-cube* by Professor A. M. Gleason of Harvard University; *Isomorph rejection in exhaustive search techniques* by Professor J. D. Swift of the University of California at Los Angeles. Chairmen at the four sessions of the Symposium were Professors Ryser, Hall, Gleason, and Dr. Hoffman respectively.

Sessions for contributed papers were held at 3:15 P.M. on Friday, and at 10:00 A.M. and 3:15 P.M. on Saturday. Chairmen at these sessions were Dr. V. E. Beneš, Professors Chester Feldman, R. P. Gosselin, E. J. Pellicciaro, Drs. G. O. Peters, F. P. Peterson. Mr. Greendlinger was introduced by Professor Wilhelm Magnus, Dr. Chen by Dr. R. A. Willoughby. Abstracts of the contributed papers appeared in the April and June, 1958 issues of the *Notices* of the Society.

The Council met on Friday afternoon, April 25, 1958.

The Secretary announced the election of the following ninety-eight persons to ordinary membership in the Society:

Mr. John Abramowich, University of California, Berkeley;
Mr. N. G. Anton, Anton Electronic Laboratories, Brooklyn, New York;

Miss Marjorie J. Bakirakis, Hood College;
Mr. P. J. Blecke, Jr., University of Chicago;
Mr. L. E. Blumenson, Columbia University;
Mr. T. E. Bonner, University of Illinois;
Mr. W. R. Brown, Republic Aviation Corporation, Farmingdale, New York;
Mr. F. J. Broughton, Research Products Company, San Santonio, Texas;
Mr. B. R. Buzby, Indiana University;
Mr. R. L. Causey, Ramo-Wooldridge Corporation, Los Angeles, California;
Dr. T. C. Chen, International Business Machines Research Laboratory, Ossining, New York;
Mr. D. R. Childs, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania;
Professor Edward M. Corson, University of Delaware;
Professor E. R. Creasy, Pennsylvania Military College;
Dr. R. B. Dawson, Jr., U. S. Department of Defense, Washington, D. C.;
Professor M. H. DeGroot, Carnegie Institute of Technology;
Mr. R. S. Dinsmore, Stanford University;
Mr. J. J. Dood, University of Illinois;
Mrs. Ellen H. Dunlap, Wayne State University;
Dr. A. B. Farnell, CONVAIR, San Diego, California;
Rev. Donald Faught, Assumption University of Windsor, Windsor, Ontario;
Dr. F. J. Fayers, California Research Corporation, La Habra, California;
Mr. R. H. Femenias, University of Miami;
Mr. P. C. Fife, New York University;
Miss Czerna A. Flanagan, U. S. Naval Ordnance Test Station, China Lake, California;
Mr. W. T. Ford, The Texas Company, Bellaire, Texas;
Mr. H. R. Foster, Kay Electric Company, Pine Brook, New Jersey;
Mr. J. M. Freeman, Massachusetts Institute of Technology;
Mr. D. C. Fried, General Electric Company, Phoenix, Arizona;
Mr. G. S. Gayron, Ramo-Wooldridge Corporation, Los Angeles, California;
Mr. R. P. Gilbert, University of Pittsburgh;
Mr. R. G. Gillespie, Boeing Airplane Company, Seattle, Washington;
Dr. S. R. Goldner, New York University;
Mr. A. G. Gross, Rensselaer Polytechnic Institute;
Dr. Bernard Grunebaum, Shell Oil Company, Houston, Texas;
Professor R. M. Gundersen, Illinois Institute of Technology;
Mr. Alexander Hachigian, University of Illinois;
Mr. J. E. Hall, Clark College;
Dr. W. A. Hijab, American University of Beirut;
Dr. Antoinette K. Huston, Rensselaer Polytechnic Institute;
Professor P. J. Hutt, Marietta College;
Professor Wolodymyr Kalyna, Ukrainian Technical Institute of New York;
Mr. Julius Kane, New York University;
Mr. Ervin Kapos, Indiana University;
Mr. R. B. Kelman, University of Illinois;
Mr. T. F. Kimes, Carnegie Institute of Technology;
Mr. W. A. Kistler, Worcester Polytechnic Institute;
Dr. R. W. Klopfenstein, R.C.A. Laboratories, Princeton, New Jersey;
Professor Heinrich Larcher, Michigan State University;
Mr. G. F. Lehman, State of Iowa Employment Service, Davenport, Iowa;

Mr. J. C. Lewis, Prairie View Agricultural and Mechanical College;
Mr. C. F. Lindblade, Wright Junior College, Chicago, Illinois;
Mr. J. R. McCarthy, College of the Holy Cross;
Mr. M. L. Madison, Colorado State University;
Dr. C. J. Marcinkowski, Polytechnic Institute of Brooklyn;
Professor K. H. Matthies, University of Cincinnati;
Mr. T. B. Mattson, U. S. Steel Applied Research Laboratory, Monroeville, Pennsylvania;
Dr. Josephine J. Mehlberg, University of Chicago;
Mr. G. H. Meisters, Iowa State College;
Mr. R. T. Mertz, International Business Machines Corporation, New York, New York;
Dr. J. M. Miller, Esso Research & Engineering, Linden, New Jersey;
Dr. A. H. Mitchell, International Business Machines Corporation, Poughkeepsie, New York;
Mr. R. B. Moler, University of Rochester;
Mrs. Ruth D. O'Dell, Brown University;
Professor Reinhard Oehme, Institute for Advanced Study;
Mr. R. C. O'Neill, Columbia University;
Mr. L. N. Orloff, Pacific Gas & Electric Company, San Francisco, California;
Mr. D. L. Patten, University of Oklahoma;
Mr. R. W. Pearson, American Machine & Foundry Company, Boston, Massachusetts;
Miss Nora Pernavs, University of Detroit;
Mr. J. D. Pincus, New York University;
Professor Boris Podolsky, University of Cincinnati;
Mr. Jehuda Rav, Hofstra College;
Mr. E. D. Rogak, University of Michigan;
Dr. J. H. Rosenbaum, Shell Development Company, Houston, Texas;
Mr. R. F. Rothschild, John Struart Inc., New York, New York;
Mr. Arthur Schlissel, Brooklyn College;
Professor H. H. Schneider, University of Nebraska;
Mr. T. A. Schoen, University of Cincinnati;
Mr. H. S. Shank, University of Chicago;
Dr. M. W. Shelly, Ohio State University;
Dr. Johann Sonner, Aeronautical Research Laboratory, WPAFB;
Mr. J. F. Spalding, General Electric Company, Utica, New York;
Professor R. L. Stanley, Washington State College;
Professor R. H. Stevens, University of Cincinnati;
Mr. A. L. Stone, Reed College;
Mr. E. J. Stovall, Jr., University of California, Los Angeles;
Mr. H. W. Sullivan, David Bogen Company, Paramus, New Jersey;
Professor T. E. Sydnor, Pasadena City College;
Mr. Y. A. Tajima, Olin Mathieson Chemical Corporation, New York, New York;
Mr. L. F. Tomko, U. S. Naval Air Development Center, Johnsville, Pennsylvania;
Dr. S. B. Townes, University of Hawaii;
Mr. R. I. VanNice, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania;
Mr. H. J. Weinitschke, Massachusetts Institute of Technology;
Mr. Clifford Wight, David Taylor Model Basin, Washington, D. C.;

Mr. P.-K. Wong, Carnegie Institute of Technology;
Mr. J. L. Wulff, Sacramento State College;
Professor R. L. Yates, University of Houston.

It was reported that the following seventeen persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Mr. G. E. Smallwood.
Harpur College: Mr. H. E. Fleming.
Johns Hopkins University: Professor D. H. Andrews, Miss Ruth L. Bari, Mr. J. T. Johnson, Mr. J. W. Marvin, Mr. R. K. Mento, Miss Janet J. Mullally, Mr. J. P. Shanahan.
Kenyon College: Mr. R. E. Mosher.
University of Missouri: Mr. N. E. Foland, Mr. C. J. Penning, Mr. C. M. Warden.
Oregon State College: Mr. J. E. McFarland, Mr. G. G. Town.
University of Rochester: Mr. W. B. Pitt.
U. S. Air Force Academy: Colonel J. W. Ault.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: *Dansk Matematisk Forening*: Mr. E. T. Poulsen; *Deutsche Mathematiker-Vereinigung*: Professor Hans-Joachim Kanold; *London Mathematical Society*: Dr. Denis Rutovitz; *Société Mathématique de France*: Mr. Jacques Collot, Dr. Jun-ichi Hano, Professor G. C. Hirsch, Professor Jean-Pierre Kahane, Dr. L. E. Motchane.

The following Presidential appointments were reported: as an Invitations Committee for a Summer Institute on Number Theory in 1959: Burton W. Jones, Chairman, P. T. Bateman, Alfred Brauer, D. H. Lehmer, Ivan Niven, A. L. Whiteman; as an Arrangements Committee for the Annual Meeting in Philadelphia in January, 1959: Emil Grosswald, Chairman, P. A. Caris, J. H. Curtiss, Robert Ellis, H. M. Gehman, W. H. Gottschalk, R. D. Schafer, G. E. Schweigert, E. T. Yang; as a member of the Employment Register Supervisory Committee for a period of one year, and as Chairman of the Committee: W. M. Hirsch; as Chairman of the Organizing Committee for Summer Institutes: R. H. Bing; as a member of the Organizing Committee for Summer Institutes for a three year period beginning July 1, 1958: Marshall Hall; as a Committee to Study New Approaches to High School Mathematics: A. A. Albert, E. G. Begle, Lipman Bers, A. E. Meder, G. B. Price, Henry Van Engen, R. L. Wilder, S. S. Wilks.

The Secretary reported that the following persons have accepted invitations to deliver hour addresses: Eldon Dyer, B. J. Pettis, Walter

Rudin, at the Summer Meeting in Cambridge; Leon Henkin, at Pomona College, November 21, 22, 1958; E. E. Floyd, at Durham, North Carolina, November 28, 29, 1958; Olga Taussky-Todd, at the April Meeting, Monterey, California, 1959; E. A. Michael, at the June Meeting in the Far West, 1959. J. M. Burgers has accepted an invitation to deliver the Gibbs Lecture at the Philadelphia Meeting in January, 1959.

The President has appointed the following persons to represent the Society: Henry J. Osner, at the dedication of the new campus at Fresno State College; Bernard Vinograde, at the Centennial Celebration of Iowa State College; W. R. Hutcherson, at the inauguration of Robert Manning Strozier as president of Florida State University; L. T. Ratner, at the inauguration of Stephen Junius Wright as President of Fisk University; Adrienne S. Rayl, at the inauguration of President Henry Stanford King at Birmingham-Southern College; Julian D. Moncill, at the inaguration of Frank Anthony Rose as president of the University of Alabama. The Delegates to the International Congress of Mathematicians in Edinburgh will be Richard Brauer, A. A. Albert, Nathan Jacobson, N. E. Steenrod, and Salomon Bochner.

The Council voted to approve the recommendation of the Colloquium Editorial Committee that Volume 29, *Foundations of algebraic geometry* by André Weil be reprinted.

The recommendation of the Applied Mathematics Committee that a symposium on Nuclear Reactor Theory be held at the Society's Eastern meeting in April, 1959, was approved.

The Council elected J. A. Dieudonné and A. M. Gleason to the editorial board of the American Journal of Mathematics to fill out the unexpired terms of André Weil and Harish-Chandra, respectively, whose resignations were effective April 1, 1958. The term of Professor Dieudonné expires December 31, 1958, and that of Professor Gleason December 31, 1959.

M. H. Heins was elected a representative of the Society to the Division of Mathematics of the National Research Council to fill out the unexpired term of S. C. Kleene, whose term expires June 30, 1959.

The Secretary reported that the officers of the Society were currently in receipt of a considerable number of requests from National Representatives and Senators to support specific legislation relating to financial or other support of science and education. The Council voted to refer such matters to the Conference Board of the Mathematical Sciences.

The Council voted that the Executive Director be asked to make a salary survey in May, 1958, similar to the survey made by the Committee on Economic Status of Teachers in May, 1957, and that the results of the survey be published in the *Notices*.

The Council authorized the President to appoint a committee to discuss the question of publishing Collected Works of Mathematicians.

R. D. SCHAFER,
Associate Secretary
JOHN W. GREEN,
Secretary

BOOK REVIEWS

Foundations of algebraic topology. By S. Eilenberg and N. Steenrod. Princeton University Press, 1952 (second printing, 1957). 15+328 pp. \$7.50.

This book has had a profound influence on the development of topology both before and after its publication. In the five years since its first printing it has become a standard textbook and reference work for anyone interested in topology.

The first course in algebraic topology is usually a difficult one for the student. He faces a mass of unfamiliar algebraic machinery whose motivation is difficult to grasp and whose applicability is appreciated only much later. Realizing this, the authors have adopted an axiomatic approach to the subject of homology theory. Starting with seven easily stated axioms relating algebra and geometry (and assuming only the basic concepts of algebra and point set topology as prerequisites) they show how many important and interesting theorems can be proved directly from these axioms. The axioms themselves are presented without motivation, but their immediate application is intended to make it easier for the student to accept them. Only after the reader has seen the power of the theory is he led into the details of the existence and uniqueness of homology theories.

In order to state the axioms the concept of *admissible category* is introduced. This is a family of pairs (X, A) of topological spaces and continuous maps f: $(X, A)\rightarrow(Y, B)$ between them which, roughly speaking, contains sufficiently many pairs and maps to state the axioms. Then a ***homology theory*** on such an admissible category consists of three functions. The first is a function which assigns to every pair (X, A) in the category and every integer q an abelian group $H_q(X, A)$. The second function assigns to every map f: $(X, A)\rightarrow(Y, B)$ in the category and every integer q a homomorphism f_*: $H_q(X, A)\rightarrow H_q(Y, B)$. The third function assigns to every pair (X, A) in the category and every integer q a homomorphism $\partial: H_q(X, A)\rightarrow H_{q-1}(A)$ (where, in the latter group, the pair $(A, 0)$ has been abbreviated to A).

The three functions H_q, f_*, ∂ of a homology theory are required to satisfy seven axioms. The first three assert the functorial (or naturality) properties of f_* and ∂. The others are: the *exactness axiom*, which relates the homology groups of (X, A), X, and A in an exact sequence; the ***homotopy axiom***, which asserts that homotopic maps induce the same homomorphism; the *excision axiom*, which asserts that $H_q(X, A)$ depends, to a great extent, only on $X-A$; and the

dimension axiom, which is a normalization condition requiring that for a single point space P the groups $H_q(P)=0$ for all $q\neq 0$. The *coefficient group* of the homology theory is then defined to be the group $H_0(P)$ where P is a single point space (it follows from the first two axioms that for any two single point spaces P, P' there is an isomorphism $H_0(P)\approx H_0(P')$).

Each of the axioms is a standard theorem of classical homology theory. The fact that these can be used as the starting point for obtaining many of the other results of homology theory is the basic idea underlying the axiomatic approach. Using only the axioms and standard facts of point set topology the authors develop the direct sum theorem and the Mayer-Vietoris sequence, calculate the homology groups of cells and spheres, and prove the Brouwer fixed point theorem, the invariance of domain, and the fundamental theorem of algebra.

Cohomology theory is developed simultaneously with homology theory. Analogous axioms are given for cohomology theory, and, when a theorem of homology theory is derived from the axioms, there is stated, at the end of the section, a similar theorem of cohomology theory whose proof is left to the reader.

The first chapter presents the axioms and some of their immediate consequences. The second and third chapters develop the homology theory of simplicial complexes from the axioms and prove the basic *uniqueness theorem* that any two homology theories with isomorphic coefficient groups are isomorphic on the category of triangulable pairs. Chapters IV, V, and VI are concerned with categories and functors, chain complexes, and formal homology theory of simplicial complexes, leading to the *existence theorem* that homology theories with arbitrary coefficient groups exist on the admissible category of triangulable pairs. Chapter VII develops singular homology theory, thus proving the existence of homology theories on the category of all pairs (X, A) and maps of such pairs.

Chapters VIII, IX, and X are concerned with Čech homology theory and its special features. This leads to another proof of the existence of homology theories. It is also shown that the Čech groups satisfy the *continuity axiom*, which states that for compact pairs (X_α, A_α) forming an inverse system then $H_q(\text{lim inf } (X_\alpha, A_\alpha)) \approx \text{lim inf } H_q(X_\alpha, A_\alpha)$. A uniqueness theorem is proved for continuous homology theories on compact pairs. Chapter XI gives some applications to euclidean spaces. Most of these applications follow directly from the axioms, but a section on manifolds uses, in addition, the continuity axiom.

Whether one agrees that the axiomatic approach is a good one for beginning students or not, there is much to recommend the book for use as a text (most suitable, perhaps, for a second year graduate course). The treatments of the singular and Čech theories are modern, complete, and quite readable by themselves. Diagrams of homomorphisms, which are used so frequently today, were first systematically used in this book, both to motivate proofs and to assist the reader in following arguments. Each chapter of the book begins with an introduction stating what the chapter covers and how the material fits into the general scheme of the book. Notes are at the end of the chapter. These discuss the historical development of the subject and its relations to other topics. References to the literature are also found in these notes. Each chapter is followed by a set of exercises. Some of these are easy and some more difficult but most of them are interesting, and the student who works his way through them will learn a great deal.

Since its publication the terminology and notation of the book has been almost universally adopted by topologists. The axioms have led to cleaner proofs of many theorems and increased their generality at the same time. In addition, the axioms have been applied to prove new results. One of the most recent of these applications is the theorem proved by Dold and Thom (C. R. Acad. Sci. Paris vol. 242 (1956) pp. 1680–1682) to the effect that the qth homotopy group of the infinite symmetric product of a polyhedron X is isomorphic to the qth homology group of X. They prove this by showing that the homotopy groups of the infinite symmetric product of X, regarded as functions of X, satisfy the axioms, whence the result follows from the uniqueness theorem for polyhedra.

The book contains no discussion of cup products or cross products. This was to be included in a projected second volume, which was to contain also a treatment of cell complexes and the practical calculation of the homology groups of such spaces. It is to be hoped that the authors have not abandoned their plan to write this second volume. Such a continuation of the present useful book would be a welcome and worthwhile contribution to the mathematical literature.

E. H. SPANIER

Vorlesungen über Himmelsmechanik. By Carl Ludwig Siegel. Springer, Gottingen, 1956. 9+212 pp. DM 29.80. Bound DM 33.

The appearance of this remarkable book is certainly one of the great mathematical events of the century. Written on the subject matter which is the mother field of modern mathematics and spar-

kling with new theorems, new proofs and the polished finesse for which its author is famous, it is a fitting monument to his genius.

This book concerns itself primarily with two central problems of celestial mechanics: Newton's three body problem (T.B.P.) and the related simpler problem of Jacobi, the restricted three body problem (R.T.B.P.). The problem in each case is to establish significant mathematical information about a set of solutions. An example of a question whose answer would be mathematically significant for the T.B.P. of sun, earth, and moon is this: Is the orbit of the moon around the earth stable?[1]

This question is not completely posed until one makes precise the definition of stability. Many different types of stability have been introduced in studies of the T.B.P., but none has had more than a very limited success. The studies centering around the stability for T.B.P. may be classified into two categories according to the methods employed: in the one the methods are purely geometric and in the other they are analytic and employ series expansions. The analytical techniques are definitely the more successful. The failure of the geometric techniques would appear to stem from their inability to exclude the topological complexities which may occur in global situations. The relative success of the analytical techniques appears to be due to the fact that they have been applied to problems which are topologically simple in that they refer to a neighborhood of a singular point or periodic solution and within that neighborhood they allow only such decompositions (by trajectories) of the phase space as can be achieved by some type of linear characterization. Although the number of topological possibilities thus allowed is quite small the number of different series expansion techniques introduced to study them is very large.

The successes of the analytical studies have always been limited. In every case where an attempt has been made to obtain conclusive results about stability by analytic means the convergence of the series employed could not be proved. It is characteristic that one could not determine whether this failure was due to a limitation of the technique employed or whether the geometric implications (which would follow from convergence) were false. Siegel's papers on celestial mechanics, prior to the publication of this book, are an attack on the T.B.P. which re-examine (at considerable depth) a large number of analytical devices introduced earlier—especially ideas of Poincaré

[1] The field is dominated by pessimists whose aim is to show that the orbit of the moon is unstable. The pessimists have a substantial lead both in the number and quality of their results, but the final outcome is hardly in sight.

and Sundman. Siegel's more recent results relate to the determination of the limitation of some of the techniques introduced by earlier workers on the T.B.P. This present book broadens studies already initiated by Siegel and others; it introduces new methods; and it sheds considerable light on G. D. Birkhoff's studies of area preserving surface transformations.

Two new results in Siegel's book deserve special mention: Siegel's Center Theorem and the Birkhoff-Lewis-Siegel Fixed Point Theorem.

Siegel's Center Theorem. Given an analytic manifold M of dimension two, a point $p \in M$, and a system of analytic ordinary differential equations defined in a neighborhood of p we shall say p is a *center* if p is an isolated singular point and in a neighborhood of p every solution, except p, is periodic. Furthermore, for an analytic arc through p with parameter σ the period $\tau = \tau(\sigma)$ of the periodic solutions depends analytically on σ. A system of analytic ordinary differential equations is said to have a *two dimensional center* at a singular point, q, if there exists a regularly imbedded two dimensional manifold through q tangent to the vector field such that the induced differential equation on the manifold has q as a center. *Theorem. Let* $x = 0$ *be a singular point of* $dx/dt = Ax + f(x)$ *an analytic Hamiltonian system* (A *constant,* $f = O(\|x\|^2)$) *with n degrees of freedom. Let A have a purely complex root* $\lambda_1 = i\omega$. *If the* $2n$ *roots of* $A \pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_n$ *are distinct and* λ_R/λ_1 *is not an integer for* $R = 2, 3, \cdots, n$ *then* $x = 0$ *is a two dimensional center.* The attempts to characterize solutions near a singular point by the characteristic roots of the linear part of the equation extend back over a hundred years. The literature on this problem is vast and it includes the names of Briot and Bouquet, Poincaré, Picard, Liapunov, Dulac and Perron. Before 1951 the hypotheses of all the results included: The smallest convex polygon in the complex plane containing the characteristic roots should not contain the origin. In 1952 Siegel showed the origin could lie inside this polygon provided the c.g. of integer valued masses distributed at the characteristic roots did not get too close to the origin *Uber die Normalform . . .* , Nachr. Akad. Wiss. Gottingen. Math.-Phys. Kl. IIa (1952) pp. 21–30. In particular the characteristic roots had to be linearly independent over the rationals. This new result is really quite surprising for it allows the c.g. of an integer valued mass distribution to be right on the origin.

The Birkhoff-Lewis-Siegel fixed point Theorem. Let T be an analytic area preserving transformation of the plane into itself for which the origin is fixed. One may determine from T a certain infinite set of real

numbers $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_R, \cdots$. Suppose $\Upsilon_1 = \Upsilon_2 = \cdots = \Upsilon_{l-1} = 0$ and $\Upsilon_l \neq 0$. Let λ and λ^{-1} be the characteristic roots of the linear part of T (expanded near $x=0$). *Theorem. If* $|\lambda| = 1$, *and* $\lambda^R \neq 1$ *for* $R = 1, 2, \cdots, 2l+2$ *then for any neighborhood* G *of the origin there is an integer* $n_0(G)$ *such that for any* $n > n_0(G)$ *there is a* $z \in G$ $(z \neq 0)$ *such that* $T^m z \in G$ *for all* m *and* $T^n z = z$.

If $\Upsilon_i = 0$ for all i the convergence problems that arise have not been settled, if they were the transformation would be analytically equivalent to a rotation, and in that event there would be a periodic point $(\neq 0)$ if and only if $\lambda^R = 1$ for some R.

The history of this beautiful theorem is strange: Siegel refers to Birkhoff's Pontifical Memoir *Nouvelle recherches sur les systems dynamiques* Memoria Pont. Acad. Sci. Novi. Lyncaei S. 3. vol. 1 (1935) pp. 85–216 (pages 132–146 are the pertinent ones). The proof there is far from complete. Moser has pointed out to us a more detailed version of this proof (a version which we believe to be also incomplete), occurs earlier in the paper of Birkhoff and Lewis *On the periodic motions near a given motion of a dynamical system.* Ann. Mat. Pura Appl. vol. 12, ser. 4, 1933 pp. 117–133. The main object of the proof of this theorem is to establish certain sharp approximations. Since Siegel has done this and Birkhoff and Lewis have not we feel the theorem should be called the Birkhoff-Lewis-Siegel fixed point theorem. That M. Morse in his mathematical biography of G. D. Birkhoff (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 357–391) does not mention this theorem as one of Birkhoff's great results is not surprising to us for although Birkhoff wrote many papers on area preserving surface transformations in dynamics the style in which they were written was such as to discourage readers. In addition, the lack of a convergence proof raised the question of whether or not these studies had any real significance. Siegel deserves a great deal of credit for his development of Birkhoff's pioneering efforts.

The material in Siegel's book is divided into three parts. I. The T.B.P., II. Periodic Solutions, and III. The Stability Problem (singular points).

The first sections of *Part* I have a short but sound introduction to Lagrangian derivatives, canonical transformations, and Hamilton-Jacobi theory, all treated in a style to which they are not accustomed. Cauchy's existence theorem is given an attractive new proof. The remaining sections are devoted to a proof of Sundman's theorem concerning the existence of a global uniformizing variable for solutions of the T.B.P. (p. 61). The discussion has been simplified by

excluding the case of a triple collision. The careful and complete treatment Siegel presents should make these results more accessible to the mathematical public, and perhaps gain new readers for Siegel's beautiful paper *Der Dreierstoss* (Annals, 1941). These results are isolated ones on the mathematical landscape and their special attraction arises in that they establish deep facts about the series expansions of solution functions by reasoning based upon geometric properties of the trajectories of the solution functions.

Part II. The first section gives Lagrange's classical triangle and straight line solutions for the T.B.P. The next three establish Siegel's Center Theorem, which is then applied to Lagrange's solutions to obtain periodic solutions not previously known. Hill's solutions in the R.T.B.P. and T.B.P. are found by new methods. Poincaré's continuity method is applied in the usual manner to the circular solutions in the R.T.B.P. After a thorough discussion of normal forms for area-preserving surface transformations near a fixed point the Birkhoff-Lewis-Siegel Theorem is established and applied to the R.T.B.P.

Part III. This part begins with a complete treatment of the function-theory center problem (Schröder's functional equation) based on earlier work by Siegel. Poincaré's center problem is solved (a new result) based on methods arising from the Siegel Center Theorem. The elementary stability theorems of Liapunov and Dirichlet are included. The next two sections are largely discussions of some of the open problems. The last section contains Poincaré's recurrence theorem.

One of the many attractive features of this work is that whenever Siegel uses complex valued linear transformations to effect a normalization of the linear part of a real system he is very careful to keep an accounting of the real solutions. Another feature is that he very frequently illustrates by means of examples the strength or the weakness of his results.

Although it is aimed primarily at celestial mechanics the results and methods of this book should be standard equipment for anyone interested in ordinary differential equations.

Mathematicians, in talking about themselves and their work, have on occasion remarked that a man's originality reaches a peak before the age of 35 and thereafter declines noticeably. In view of the originality and power Siegel exhibits in this book it would appear that a mathematician's scientific life span has been extending itself at a very reassuring rate.

STEPHEN P. DILIBERTO

Integral equations. By F. G. Tricomi. New York, Interscience, 1957. 8+238 pp. $7.00.

This excellent textbook on integral equations was written to give an adequate introduction of the subject to those who require a knowledge of it in mathematics or in its applications. With a basic knowledge of the theory of the Lebesgue integral and the theory of functions of a complex variable, this book may be read with profit.

The first three chapters are devoted to the so-called regular theory. Here we are given a careful and leisurely discussion of the Volterra equation, the regular Fredholm equation and equations with symmetric kernels. Such a program may sound traditional but an examination of the text proves otherwise. For one thing, the text is liberally peppered with physical problems which lead to such integral equations. Another feature reveals to us the relation between linear differential and integral equations. When further background material in analysis is required, Professor Tricomi supplies the reader with a discussion of the important features and provides references. Many of the historical remarks are of great interest. In short, we have an interesting and sparkling account of the regular theory of linear integral equations which should hold the attention of a serious worker.

The last chapter deals with singular and non-linear integral equations. Integral equations of the convolution type (Abel, Picard and Wiener-Hopf) are merely mentioned since they rightly belong to the realm of Fourier transform theory, a chapter of the subject which the author does not choose to discuss in detail. The main class of singular, linear, integral equations which is treated is of the Cauchy type, and with it, the Hilbert transformation. Carleman's function theoretic method for handling equations of this type is discussed and it is pointed out (at last!) that this work preceded that of Vekua, Mikhlin, etc. We should be thankful to Professor Tricomi for having corrected the misconception which has crept into the literature. The chapter closes with some remarks about non-linear integral equations. A notable feature of this chapter is the care with which one is introduced to the study of singular equations.

There are two appendices, which are intended to round out the text—one on systems of linear equations, the other on Hadamard's theorem on determinants. Some exercises, as well as a small bibliography are provided.

The text may be heartily recommended to pure and applied mathematicians.

Albert E. Heins

Integral equations and their applications to certain problems in mechanics, mathematical physics and technology. By S. G. Mikhlin. Translated from the Russian by A. H. Armstrong. New York, Pergamon Press, 1957. 12+338 pp. $12.50.

This is a substantial but not an exciting account of the theory of integral equations of the regular and singular type with applications to the physical sciences. The book is divided into two parts, the first third giving the theory and the second two-thirds discussing the applications.

The regular cases, which include those equations whose kernels may be made regular by a finite number of iterations, are studied by the classical methods of Fredholm, Hilbert and Schmidt. Various devices for determining characteristic values are considered and some numerical examples are given. In the last quarter of this section a brief account is given of the type of singular integral equation which the author needs in some of his applications. That is, this final chapter on integral equations discusses those equations whose kernels are of the Cauchy or Hilbert type with the related function theoretic methods required to obtain solutions for these equations. This account is readable and the reader will not be annoyed with the masses of special detail which other writers have attempted to supply on this topic. The stage has now been set to treat boundary problems of the Hilbert-Riemann type which arise so often in the applied fields.

An interesting assortment of problems involving integral equations which arise in the mechanics of continuous media and which come from such diverse equations as Laplace's, the biharmonic, the wave and the diffusion equation (all in two dimensions) is discussed with the machinery developed in the first part of the book. Like his compatriot, Muskhlishvili, Mikhlin refers to unusual problems in wave motion which integral equations can handle and then fails to provide the reader with some of the important details.

ALBERT E. HEINS

Elements of the theory of functions and functional analysis, Vol. 1, *Metric and Normed Spaces.* By A. N. Kolmogorov and S. V. Fomin. A translation by L. Boron of *Elementy Teorii Funkciĭ i Funkcional'nogo Analiza.*, I. *Metričeskie i Normirovannye Prostranstva.* Rochester, Graylock Press, 1957. 9+129 pp. $3.95.

This little book, the first in a projected series by the same authors, is a textbook prepared from material presented at the Moscow State University and could serve as a text for, or as a welcome adjunct to,

a semester (or quarter) course in functional analysis for beginning graduate students. The following list of chapter headings and principal topics will indicate its scope.

Chapter 1, devoted to set theory, contains no surprises. We begin with a brief discussion of the concept of set and progress through the usual sequence of topics, viz., set operations, maps, cardinal numbers, equivalence relations, equivalence class decompositions and their associated projections. There are several illuminating examples and exercises. Order relations are not mentioned. Chapter 2 develops the theory of metric spaces and is noteworthy in several respects, both for what is included and for what is not. We have, of course, sequences and limit points, closed and open sets, continuity and isometry, separability, completeness and completion. There is a special section devoted to the structure of open sets on the real line. The usual discussion of uniformity is simply omitted; the concept of a topological space is introduced but plays no further role (non-metrizability is illustrated in a 2-point space); the category theorems are deferred; G_δ's do not put in an appearance. In exchange for these more familiar discussions, we are offered several interesting and relatively unfamiliar ones. As early as possible, the authors prove the theorem that a properly contracting map on a complete metric space has a unique fix-point, and proceed, at once, to exploit the principle to the hilt in a number of excellently chosen examples, each worked out in detail. The examples include, most notably, successive approximation to the solution of a system of linear equations with sufficiently dominant main diagonal, integral equations with continuous kernel, and the Picard-Lipschitz existence theorem. Chapter 2 also contains an unusually thorough treatment of compact sets in metric spaces; included is the criterion for the compactness of a set in a $C(X, Y)$ (Arzela's theorem), also illustrated in the context of differential equations—this time Peano's existence theorem is proved in detail. The chapter closes with a long and interesting, if not always quite lucid, discussion of arc-length in a metric space, including a proof of the existence, under suitable conditions, of a shortest arc between two points. Chapters 1 and 2 together occupy slightly over half of the book and seem to the reviewer to comprise by far the most stimulating and successful portion thereof.

Chapter 3 takes up the theory of normed spaces. After a brief section on convex sets, the usual topics surrounding the notion of linear functional are developed. The Hahn-Banach theorem is stated generally but proved in the separable case only. The dual space, the second dual, the notions of reflexivity and weak convergence of

sequences of elements and functionals are introduced and briefly illustrated and discussed. Direct sums, quotient spaces, bases are not mentioned. Neither are the weak topologies as such. (Neither, mysteriously, is the completion of a normed space, though the completion of an arbitrary metric space was constructed in Chapter 2.) The chapter closes with a section on linear transformations which develops the basic facts concerning boundedness, algebraic combinations, adjoints and inverses of operators. After Chapter 3 is interpolated a five page appendix giving an illuminating account of the elements of the theory of generalized functions (distributions). The fourth and final chapter on linear operator equations is quite brief. Attention is sharply focused on the compact case. The Fredholm theorems are treated, and it is shown that a usefully broad class of mildly discontinuous kernels defines compact operators. In contrast with the wealth of illustrative material in Chapter 2, the examples in Chapters 3 and 4 become gradually scarcer and more routine until, at last, they peter out altogether; the Fredholm theorems are unaccompanied by either example or exercise.

Attention has already been called to the absence of some topics that would be thought by many to form a natural or even an indispensable part of a book on the "elements" of functional analysis. The list could be lengthened. Lebesgue integration is not once mentioned (though the foreword promises its appearance in a later volume). Neither is Stieltjes integration. (The only linear functionals on $C(X)$ that appear are atoms.) Even such a work horse as the Banach-Steinhaus theorem is not to be found. These omissions of more or less standard material are not cited as errors of judgment or faults of the book, but rather to indicate its spirit. Indeed, its title is one that would mean different things to different mathematicians. The authors of the present book have attempted to assemble, in compact, consequential, and conceptually unified form, those things that will prove most useful to the reader in subsequent encounters with the methods of functional analysis in applied mathematics; and they judge accordingly that, say, the Fredholm alternative and the Neumann series are elemental, while a constructive characterization of the L_2 completion of the space of continuous functions is an interesting sidelight that may safely be postponed. This point of view is, regrettably, foreign to much current instruction in functional analysis, at least in this country, and the present book is quite unlike any text heretofore available in English. (On the other hand, it is quite close, both in spirit and content, to the earlier and more extensive treatise with the same title by Sobolev and Lusternik.) Its transla-

tion is opportune and will be welcome to students of mathematical physics as well as to students of mathematics.

There are several errors in the book—some troublesome, some not. In the latter category, it is perhaps worth mentioning that the assertion about the reflexivity of $\bar{E}$ (p. 90, line 13) should, of course, be restricted to complete spaces. (Even so restricted the statement is unclear at best; R. C. James has shown that E and $\bar{\bar{E}}$ may not be "distinct" even if E is not reflexive.) The proofs offered for two of the main Fredholm theorems (Theorems 2 and 3, pp. 119–120) are inadequate, but the errors betray themselves at once and the reader will have no trouble finding correct proofs elsewhere. On the other hand, the proposed metrization of the weak*-topology on the entirety of an infinite dimensional dual space (p. 95) is likely to confuse the student, as is the mistaken assertion (p. 92) that linear combinations of atoms are uniformly dense in the dual of $C[0, 1]$. (They are weakly dense, of course.) The special role of 0 in the spectrum of a compact operator is overlooked (p. 116 and again p. 120). The proof (p. 65) that a function uniformly close to a function of unbounded variation has itself unbounded variation is (necessarily) fallacious. Finally, the reviewer disputes the assertion that the equivalence relation introduced on p. 68 is obviously transitive.

With few exceptions the translation is *verbatim*, and is for the most part felicitous. A very few liberties have been taken with the text, to correct an oversight or to bring a definition into agreement with custom; the usual Bunyakovski→Schwarz transformation has been carried out; topicalities have been removed, so that, for instance, the "set of all automobiles in Moscow" has become the "set of all automobiles in a given city." On the whole one inclines to object that the translation is too literal. The errors mentioned above have been carefully preserved, and, on at least one occasion, even a misprint is reproduced. The translator has enlarged the list of references to include a number of books in English and has provided lists of symbols, of theorems and of definitions (the last sadly incomplete).

In the original version the authors employed, in a familiar fashion, two sizes of type; one, regular, for the main thread of the book, and a smaller one reserved for remarks, examples etc. ancillary to the main development and not necessary to its understanding. Shorter asides were relegated to footnotes. In the present edition the distinction between type faces has been completely ignored and all the footnotes have been thrust abruptly into the text. This, of course, robs the reader of the authors' opinion of the relative importance of the various parts of the book and results in making it considerably harder

to read. What were, before, carefully articulated discussions have become, somehow, amorphous. It is hard to imagine that the saving in cost is worth the loss incurred.

ARLEN BROWN

The topology of fibre bundles. By Norman Steenrod. Princeton, Princeton University Press, 1951, second printing, 1957. 8+227 pp. $5.00.

This second printing of Steenrod's well known book differs from the first only in the addition of an appendix which describes progress in the field between 1951 and 1956. The rate of progress has been very high indeed, so that even this appendix requires amendment to bring it up to date.

Today as in 1951 the term "fibre space" is ambiguous: the definitions due to Serre, Hu, and Hurewicz all have a good claim to the title. However, thanks to Steenrod's book, the "fibre bundle" is now a familiar and well defined object. In the applications of topology to differential geometry, and lately also to algebraic geometry, the fibre bundle is a tool of fundamental importance.

The following is a brief description of the book. Part I sets a foundation for the study of fibre bundles. Concepts such as cross-section, bundle map, induced bundle, and principal bundle are defined and studied with the author's characteristic thoroughness. A number of important examples are considered: tensor bundles, covering spaces, the principal bundle over a coset space, and so on. It is shown that any cross-section of a differentiable bundle can be approximated by a suitably differentiable cross-section. As is pointed out in the appendix, many of the theorems of Part I can be technically improved. However this remains the best presentation of the subject matter which is available.

Part II studies the homotopy theory of bundles. The homotopy sequence of a bundle is defined; a classification theorem for bundles over the n-sphere is proved; the theory of universal bundles is developed; and the Hopf fiberings are studied. A number of results about the homotopy groups of spheres and other standard manifolds are obtained. However this last material has been outclassed by subsequent developments in the field. [For recent work see: R. Bott, *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 43 (1957) pp. 933–935; F. Adams, *On the structure and applications of the Steenrod algebra*, Comm. Math. Helv., to appear; and G. F. Paechter, *The groups* $\pi_r(V_{n,m})$ (I), Quart. J. Math. Ser. (2) vol. 7 (1956) pp. 249–268.] Finally the tangent bundle of the n-sphere is studied. [For recent work see forthcoming papers by Kervaire

(Proc. Nat. Acad. Sci. U.S.A.); by Bott and Milnor (Bull. Amer. Math. Soc. vol. 64 (1958) pp. 87–89.; and by James (Proc. London Math. Soc.).]

Part III, *The cohomology theory of bundles*, contains an excellent exposition of obstruction theory, using bundles of coefficients. This theory is applied to the study of Stiefel-Whitney classes; however in view of the remarkable developments due to Thom and Wu, this approach to Stiefel-Whitney classes has become more or less obsolete. The next section studies the problem of whether or not a given n-manifold possesses a continuous field of quadratic forms of signature k. (I.e. does the tangent bundle split into the "Whitney sum" of a k dimensional vector space bundle and an $(n-k)$-dimensional vector space bundle?) Finally the Chern classes of a complex analytic manifold are considered.

The following is an illustration of the way in which material in Steenrod's book has led to important research. The problem of classifying sphere bundles over spheres occupies only six pages of the book. This led to work by James and Whitehead on the problem of classifying the resulting total spaces as to homotopy type. [See Proc. London Math. Soc. vol. 4 (1954) and vol. 5 (1955).] Hirzebruch used S^2-bundles over S^2 to show that the same differentiable manifold may possess several distinct complex structures. [Math. Ann. vol. 124 (1951).] The reviewer used S^3-bundles over S^4 to show that the same topological manifold may possess several distinct differentiable structures. [Ann. of Math. vol. 64 (1956).] Several authors have used S^3-bundles over S^4 to show that manifolds of the same homotopy type may be distinguished by their Pontrjagin classes. [R. Thom, Ann. Institut Fourier, Grenoble vol. 6 (1955–1956) p. 81; I. Tamura, J. Math. Soc. Japan vol. 9 (1957); and N. Shimada, Nagoya Math. J. vol. 12 (1957).]

In conclusion, this book remains a must for students of topology and geometry.

JOHN MILNOR

Differential equations: Geometric theory by S. Lefschetz. New York, Interscience Publishers, 1957. 10+364 pp. $9.50.

The present work is modestly referred to by the author as an extension of his previous Annals of Mathematics Study, *Lectures on differential equations*. Actually, the additional material included makes this a book which differs in character from its predecessor. The *Lectures on differential equations* was strictly a textbook for, say a first year graduate course in ordinary differential equations. The present volume, while it contains some introductory material (Chap-

ters I–III), is primarily a treatise on the two aspects of differential equations theory which the Princeton Group in Differential Equations, founded by Mr. Lefschetz, have found most interesting. These topics are stability theory (Chapters IV–VIII) and two-dimensional systems (Chapters IX–XII).

The book thus serves a number of purposes: It can be used as a starting point for mathematical investigation in stability theory or two-dimensional systems. It may be useful to engineers and physicists who wish to inform themselves of the current state of knowledge in stability theory. It can be used as a text for a one-term course using Chapters I–III plus selected topics from the last four chapters. Finally, it is possible in a two-term graduate course in ordinary differential equations to spend the first term on linear equations and the second term on nonlinear equations. In this case it would be possible to use the book by Coddington and Levinson for the first term and the new book by Lefschetz for the second term.

As indicated, the book divides into three parts. The first three chapters and the appendix consist of introductory material. Chapter I and the appendix review the theory of functions of several variables as well as the topological and algebraic background material. Since the emphasis in the whole volume is on analytic systems the Weierstrass Preparation Theorem is introduced at this point. It later becomes an important tool in the study of singular points. The topological tools include metric spaces, differentiable manifolds, the index of a circuit, the Brouwer Fixed Point Theorem and the Poincaré Index Theorem. The review of linear algebra includes Jordan Canonical Forms and a discussion of logarithms of matrices.

In Chapter II the basic existence and uniqueness theorem is proved for the differential equation $dx/dt=f(x, t)$, where f is continuous and satisfies a Lipschitz condition. The solutions are then shown to be continuous functions of the initial conditions; they are differentiable (analytic) functions of the initial conditions if f is differentiable (analytic).

Chapter III is an introduction to linear differential equations. It has been previously pointed out that this book tends primarily towards nonlinear equations. This is perhaps best illustrated by the fact that the author motivates the study of linear differential equations by showing that linear differential equations may arise as the equation of first variation of non-linear equations. In line with this general outlook, linear differential equations are held to a minimum (20 pages). The author discusses the existence and uniqueness theory, the facts regarding the vector space of solutions, and the adjoint

equation and its use in the solution of non homogeneous equations, linear systems with constant and periodic coefficients make up the rest of the chapter.

The most important part of this book is probably the section on stability, which is an excellent survey of the field. Not only is the choice of topics unusually good, but this section introduces the American reader to some recent work of Russian mathematicians, which has not been widely available. The section starts out, in Chapter IV, with the basic definitions of stability theory. The definitions are in line with those used by Malkin, Massera, and Antosiewicz. In Chapter V the basic stability theorem for the equation $dx/dt=Px+q$ is proved. Moreover, it is shown that a suitable process of successive approximations will tend towards a solution of this equation if the characteristic roots of P all have negative real parts and if q is suitably restricted. The rest of this chapter deals primarily with analytic systems and is built around the Liapunov Expansion Theorem. This Theorem describes a series solution of the nonlinear equation if the characteristic roots of P have negative real parts and are well behaved in the sense of Liapunov. The terms of this series are essentially bounded functions of t multiplied by certain exponentials which depend on the characteristic roots of P.

Chapter VI discusses the very important direct method of stability theory. This method consists of constructing something like a potential function near the trajectory under study. The author then develops some recent work by Persidski, Malkin, and Dychman. The theorem of Dychman can be used to study a two-dimensional nonlinear system where the coefficient matrix of the first-order terms has a zero root.

Chapter VII studies the stability properties of the system $dx/dt=P(t)x+g(x, t)$. In this case the stability properties of $dz/dt=P(t)z$ no longer determine the stability properties of the nonlinear system. The author uses a Theorem of Perron's to simplify the problem. This theorem of Perron's says roughly that the stability of the system is not changed if $P(t)$ is replaced by a similar triangular matrix. A number of sufficient conditions for the stability of the zero-solution of $dx/dt=P(t)x+g(x, t)$ are developed. The author proves that the various criteria for $dz/dt=P(t)z$ known as Persidski Conditions, Perron Condition, existence of a quadratic Liapunov Function are equivalent. They all imply stability for the non-linear system.

Chapter VIII deals with the stability theory of periodic systems. The basic theorem relating the characteristic exponents to asymptotic stability is proved. A special discussion of the autonomous system

which is not covered by the general theorem follows. Quasi linear systems and complete families of periodic solutions are also included in this chapter.

The last few chapters are a very fine exposition of two-dimensional systems of differential equations. Chapter IX starts with an investigation of simple critical points. The index of simple critical points is computed. The chapter ends with a study of the Technique by which a differential equation can be extended from the Euclidean Plane to the Projective Plane, a method which has been neglected since Poincaré. Chapter X investigates general critical points. It is shown that for analytic systems the index of a critical point is equal to $1+1/2$ (number of elliptic section − number of hyperbolic section). The notion of the limit sets of a trajectory is taken up next. It is shown that the limit sets of a trajectory fall into four mutually exclusive categories. Critical points with a single zero characteristic root and structural stability are other highlights of this chapter.

Chapter XI discusses the equation $d^2x/dt^2+f(x)dx/dt+g(x)=e(t)$. We are concerned here with proving the existence and uniqueness of periodic solutions. The author exhibits a variety of techniques for accomplishing this goal, all taken from recent literature. Chapter XII studies the perturbation theory of second order differential equations. Sufficient conditions for the existence and uniqueness of periodic solutions of the perturbed systems are found. The stability of these periodic solutions is investigated. Other topics discussed in this chapter are the Stroboscopic method of Minorski, relaxation oscillation, and the stability zones of the Mathieu Equation.

Summarizing, this is a very interesting book containing a wealth of material. This work should be useful to a variety of mathematicians and physicists.

FELIX HAAS

Einführung in die transzendenten Zahlen. By. Th. Schneider. Berlin-Göttingen-Heidelberg, Springer, 1957. 7+150 pp. DM 21.60. Bound DM 24.80.

A complex number α is said to be algebraic or transcendental (over the rational numbers) according as it is or is not a root of an equation of the form $a_0x^n+a_1x^{n-1}+ \cdots +a_n=0$, where $a_0, \cdots, a_n$ are rational numbers. The distinction between these two kinds of numbers was recognized at least as early as Euler (1744), who asserted that the logarithm to a rational base of a rational number must be either rational or transcendental. This was a bold conjecture indeed, since at that time no example of a transcendental number was known,

and no method was available even of constructing "artificial" transcendental numbers, much less of dealing with the question of the transcendence of a number given beforehand. It was a full century later that J. Liouville found a useful characterization of a class of transcendental numbers, by means of which such numbers could be constructed. He proved that if α is algebraic of degree n (that is, satisfies an irreducible polynomial equation of degree n), then there is a constant $c>0$ such that the inequality

$$(1) \qquad |\alpha - p/q| < c/q^{\mu}$$

has no solutions in integers p and q, if $\mu = n$. Hence α is transcendental if for fixed c and for every positive number μ the inequality (1) has a solution with $q>1$, and it is not difficult to see that the number $\alpha = \sum_1^\infty 2^{-k!}$, for example, has this property.

After Liouville there were two other notable results in the nineteenth century: in 1873 C. Hermite proved that e is transcendental, and nine years later F. Lindemann generalized Hermite's theorem in such a way that the transcendence of π became apparent from the relation $e^{\pi i} = -1$. However, even as recently as 1930 it would have been accurate to say that there was no real theory of transcendental numbers, but only a scattered set of results. Since that time, matters have improved considerably, and the present book demonstrates conclusively that there is now a respectable battery of general methods available both for unifying older work and for attacking new problems.

For the creation and development of these new methods, we are indebted primarily to four mathematicians: A. O. Gel'fond, K. Mahler, T. Schneider and C. L. Siegel. It is apparent on every page of Schneider's book that this is the work of a master of his field. It is evident too that the proofs have been reworked, simplified, and polished until they are as nearly elegant as this kind of estimative mathematics can be made. This has its advantages and disadvantages, of course; for one already familiar with the details of arguments similar to those omitted, the book is a pleasure to read, but to the beginner the gaps may be troublesome to fill. One might also wish that the author had included a final chapter on isolated results not fitting into the broad pattern of the book, and simultaneously completed the bibliography, but as it stands the book is a substantial contribution to the literature of the subject. (There are only two other modern books on transcendental numbers, by Siegel and Gel'fond; neither is as comprehensive as Schneider's.)

The first chapter is devoted to generalizations and improvements

of Liouville's theorem. After intermediate work by A. Thue, Siegel, and F. J. Dyson, it has recently been proved by K. F. Roth that the theorem surrounding inequality (1) above remains true if μ is any constant larger than 2. The author combines Roth's method with earlier work by himself and Mahler to prove the following theorem, which reduces to Roth's in case $b=\eta=1$: *Let α be an irrational algebraic number, and let b be a positive integer. Let $\{p_\nu/q_\nu\}$ be an infinite sequence of rational numbers with $q_{\nu+1}>q_\nu>0$, where the denominators q_ν can be represented as products $q_\nu=q'_\nu\cdot q''_\nu$ of integers such that q''_ν is a nonnegative integral power of b. Put*

$$\eta = \limsup_{\nu\to\infty} \frac{\log q'_\nu}{\log q_\nu} \cdot$$

Then if $\mu>\eta+1$, the inequality (1) *has only finitely many solutions p/q from the sequence $\{p_\nu/q_\nu\}$*. (It might be worth mentioning that Schneider's theorem, in turn, is contained in one by D. Ridout, to appear soon in Mathematika. Using the latter, Mahler has settled a basic question concerning Waring's problem.) The above theorem is applied to demonstrate the transcendence of values of a variety of series, perhaps the most interesting being of a type considered by Mahler twenty years ago, which includes the decimal fraction $0.12345678910111213\cdots$.

Chapter II is primarily concerned with applications of the following theorem: *Let $f_\nu(z)$* $(\nu=1, 2)$ *be meromorphic functions with the following properties*:

(a) *Each $f_\nu(z)$ can be represented as a quotient of entire functions of orders at most μ;*

(b) *Each $f_\nu(z)$ satisfies a differential equation of the form $f^{(k)}(z) =P_\nu(f(z), f'(z), \cdots, f^{(k-1)}(z))$, where P_ν is a polynomial with coefficients in an algebraic number field K of degree s;*

(c) *The $f_\nu(z)$, and all their derivatives, assume values in K at the distinct points $z_0, \cdots, z_{m-1}$.*

Then if $m>(2\mu+1)(3s-1/2)$, there must be an algebraic relation between $f_1(z)$ and $f_2(z)$.

This theorem is a descendant (albeit some generations removed, mathematically) of one due to G. Pólya, to the effect that 2^z is in a certain sense the least rapidly increasing integral transcendental function which assumes rational integral values for $z=1, 2, \cdots$. Gel'fond was the first to use such theorems for transcendence problems. Choosing $f_1(z)=z$, $f_2(z)=e^{\alpha z}$ ($\alpha\neq 0$, algebraic) and $z_\lambda=\lambda$ ($\lambda=0, 1, 2, \cdots$), it follows immediately, since the conclusion of

the above theorem is false, that e^{α} is transcendental. With $f_1(z)=e^z$, $f_2(z)=e^{\beta z}$ (β irrational) and $z_\lambda=\lambda \log \alpha$ ($\alpha\neq 0$, 1) it is equally easy to prove the generalization of Euler's conjecture given by Hilbert as the seventh in his famous list of problems: if α is not 0 or 1, and β is irrational, then at least one of α, β, α^{β} is transcendental. Many statements about values of elliptic functions, elliptic integrals and modular functions are also deduced, and mention is made of generalizations involving functions of several complex variables and leading to theorems about abelian functions and abelian integrals.

In Chapter III the mutually related classifications of complex numbers due to Mahler and J. F. Koksma are developed. In each scheme, every complex number belongs to one of four classes, according to the exactness with which it can be approximated by algebraic numbers. Elements from different classes are algebraically independent; one class consists of the algebraic numbers themselves; another, containing generalized Liouville numbers, is uncountable but of measure zero; the third may be empty; and the fourth contains almost all real numbers, and almost all complex numbers. The relation between the two classifications is worked out in detail, and the best result known to date (due to the reviewer) concerning a measure-theoretic conjecture advanced by Mahler, is proved.

If ξ is a transcendental number and $P(z)$ is a nonzero polynomial with rational integral coefficients, then $P(\xi)\neq 0$. How close $P(\xi)$ can be to zero depends of course on how complicated $P(z)$ is, that is, on how large are the degree n and the height H—the maximum of the absolute values of the coefficients in $P(z)$. Given ξ, a function $T(n, H)$ such that $|P(\xi)| \geqq T(n, H)>0$ for all nonzero polynomials P is called a measure of transcendence for ξ. The fourth chapter is devoted to the derivation of measures of transcendence for e (Mahler) and α^{β} (Gel'fond); e.g., it is shown that

$$|P(e)| > H^{-n-cn^2 \log (n+1)/\log \log H}$$

for an appropriate constant c and for all polynomials P with $n\geqq 1$ and $H>H_0(n)$. Moreover, the following theorem, which is an essential improvement of one due to P. Franklin, is proved: *Let α and β be two complex numbers, with $\alpha\neq 0$, 1 and β irrational. Let η_1, η_2, η_3 be zeros of polynomials of degrees $\leqq n$ and heights $\leqq H$. Then if $k>5$, the inequality*

$$\max (|\alpha-\eta_1|, |\beta-\eta_2|, |\alpha^{\beta}-\eta_3|) < e^{-\log^k H}$$

is not solvable with arbitrarily large H. This theorem has apparently not been published before.

The final chapter, on algebraic independence of transcendental numbers, contains an exposition of Siegel's method, by means of which it can be shown, under suitable circumstances, that algebraically independent solutions of a system of linear differential equations have algebraically independent values for algebraic argument. The general framework of the method is set out very clearly, and applications are made to (a) Lindemann's theorem, that if $\alpha_1, \cdots, \alpha_n$ are algebraic numbers, linearly independent over the rationals, then $e^{\alpha_1}, \cdots, e^{\alpha_n}$ are algebraically independent over the rationals; and (b) Siegel's theorem, that if $x_0 \neq 0$ is algebraic, the values $J_0(x_0)$ and $J_0'(x_0)$ of the Bessel function are algebraically independent over the rationals.

The book ends with an interesting list of open problems which the author considers both important and attainable.

W. J. LeVeque

Applied analysis. By Cornelius Lanczos. Englewood Cliffs, Prentice Hall, 1956. 20+539 pp. $9.00.

This is a systematic development of some of the ideas of Professor Lanczos in the field of applied analysis, rather than another textbook. Instead of the adjective "applied," the author suggests that "parexic"[1] would be more appropriate. Parexic analysis lies between classical analysis and numerical analysis: it is roughly the theory of approximation by finite (or truncated infinite) algorithms. There is, therefore, some overlap with what is being called elsewhere "constructive theory of functions." However, this book is a very personal one, and has been influenced by the varied experiences of the author: in regular academic work, in government service, in industry, and now at the Dublin Institute for Advanced Study.

This is not a book for beginners: however an advanced class of mathematically inclined engineers (or physicists) or numerically inclined mathematicians would be rewarded by working through it critically. The various topics covered: algebraic equations, matrices and eigenvalue problems, large scale linear systems, harmonic analysis, data analysis, quadrature and power expansions, are all illustrated by numerical examples worked out in detail. There is a useful collection of tables, in particular a table of Laguerre polynomials, to facilitate the study of electrical networks by means of Laplace transforms.

Practicing numerical analysts will be glad to have available up-to-

[1] The word is derived from the Greek para, almost, ek, out of.

date accounts of the various techniques they associate with the author such as: Chebyshev approximation, the τ-method, minimized-iteration and spectroscopic eigenvalue analysis. They will find many uses of these methods not only in their day-to-day problems, but also in their research activities.

JOHN TODD

Die Berechnung der Klassenzahl Abelscher Körper über quadratischen Zahlkörpern. By Curt Meyer. Berlin, Akademie-Verlag, 1957. 9+132 pp. DM 29.

The class number formula for an abelian extension of the rational field, originally given by Dirichlet and Kummer for a quadratic or cyclotomic field respectively, is certainly one of the most beautiful results in classical number theory. In the present book, the author studies the problem of establishing a similar formula for the class number of a finite algebraic number field K which is contained in an abelian extension of a quadratic field F. When K itself is an abelian extension of F, certain results in this direction have been already obtained by Dedekind, Fueter and Hecke. But the author gives here a complete and systematic solution of the problem including all these previous results.

With the introduction explaining the historical background of the problem, the book is divided into three parts with the following titles: I. Algebraic, arithmetic and analytic foundations, II. Kronecker's "Grenzformeln" for the L-functions of the ring- and Strahl-classes in quadratic fields, and their application on the summation of L-series, III. Class number formulae. Of these, the most essential one is the second part which occupies about two thirds of the whole book. Here the author carefully carries out the summation of L-series with classical technique, considering several cases separately according as F is real or imaginary and, also, according to the nature of the conductor of the character in the given L-series. The actual computation is not very simple, but it is neatly given in every detail.

Though the book deals exclusively with a rather special topic, the final result, namely, the class number formula for K, seems to have wide implications in algebraic number theory, suggesting many important problems in the field. In case F is imaginary, the class number of K is expressed in terms of certain singular values of the functions which are familiar in the theory of elliptic modular functions, and this naturally suggests that there exists a deep relation between the class number formula and the theory of complex multiplication which is yet unknown to us. On the other hand, if F is real, the class

number of K is expressed by means of logarithmic integrals of the kind of functions used in the imaginary case and the analogy with the imaginary case suggests that functions related with these integrals might be used in constructing abelian extensions over F. Finally, as Hecke noticed, it may be also quite interesting if we can find from these class number formulae some special kind of units in K which correspond to the circular units in a cyclotomic field.

In the present book, none of these problems is discussed. But, as the author hopes, the book will certainly serve as the foundation for those who want to work on this stimulating territory where many branches of mathematics act on each other to solve difficult problems.

KENKICHI IWASAWA

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